Generalised Aldous' problem on spectral gaps

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Outline

Aldous' conjecture

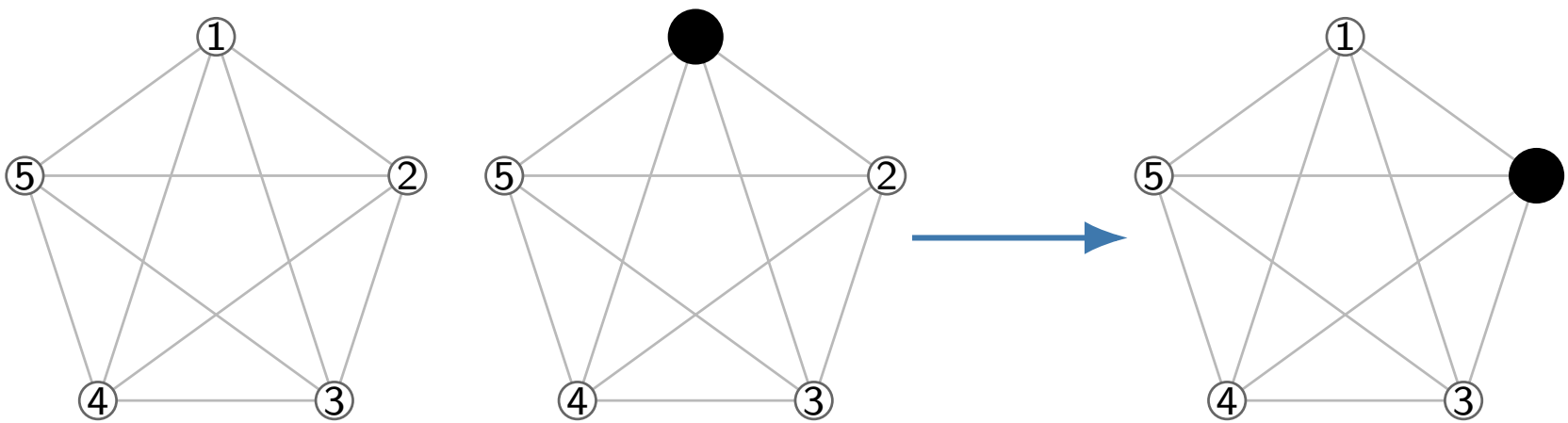
Three interpretations and generalisations

Full-flag Johnson graphs

Huang-Huang-Cioabă's conjecture

Continuous-time random walk

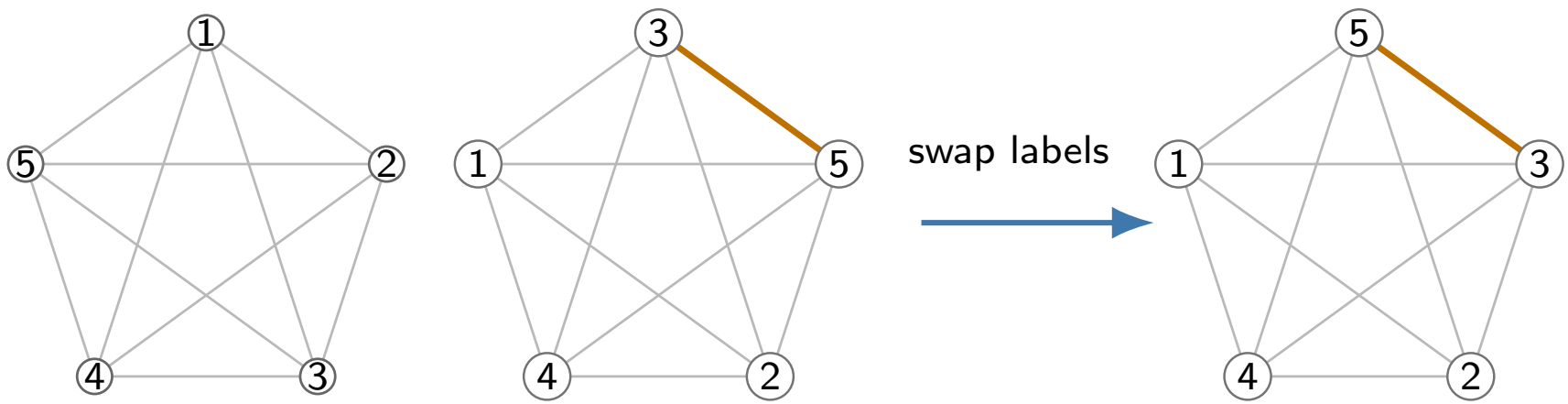
- Fix a connected weighted graph Γ on $[n]$, with edge-weights $w_{ij} = w_{ji} \geqslant 0$.
- A single particle jumps along an edge $\{i, j\}$ at rate w_{ij}.
- The state space is just the vertex set $[n]$.



A transition from the state 1 to the state 2.

Continuous-time interchange process

- Put one labelled particle at each vertex of Γ.
- When the edge $\{i, j\}$ rings at rate w_{ij}, the two endpoint labels are swapped.
- A state records the whole labelling; equivalently, the state space is S_n.



The highlighted edge swaps its two endpoint labels.

Aldous' spectral gap conjecture

Aldous' conjecture 1992

$$\mathrm{gap}_{RW} = \mathrm{gap}_{IP}$$

- The random walk lives on n states, while the interchange process lives on $n!$ states.
- So the statement is surprising: a much larger chain mixes at exactly the same asymptotic rate.

The same phenomenon in three fields

This conjecture lies at the interface of algebra, probability, and graph theory.

- **Probability:** the random walk and the interchange process have the same spectral gap.
- **Graph theory:**

$$\lambda_2\big(\mathrm{Cay}(S_n, \Sigma)\big) = \lambda_2\big(\mathrm{Sch}(S_n, [n], \Sigma)\big).$$

- **Representation theory:** the second eigenvalue of the Cayley graph is attained by the standard representation.

That is to say, we can treat this problem in three different ways.

A graph-theoretic reformulation

The edge-weights w_{ij} on Γ can be transferred to the transpositions $(ij) \in S_n$. Let $T = \{(ij) : w_{ij} > 0\} \subseteq S_n$.

Random walk on Γ
State space: $[n]$
Graph model: $\mathrm{Sch}(S_n, [n], T)$

Interchange process
State space: S_n
Graph model: $\mathrm{Cay}(S_n, T)$

Here $\mathrm{Sch}(S_n, [n], T)$ has vertex set $[n]$, and a transposition $(ij) \in T$ acts on a point $k \in [n]$ by $k \mapsto (ij)(k)$.

Aldous' conjecture in graph-theoretic form

The two weighted graphs above have the same Laplacian spectral gap: $\mathrm{gap}\big(\mathrm{Cay}(S_n, T)\big) = \mathrm{gap}\big(\mathrm{Sch}(S_n, [n], T)\big)$.

A representation-theoretic reformulation

The natural permutation action of S_n on $\mathbb{R}^n$ decomposes as

$$\mathbb{R}^n = \underbrace{\langle \mathbf{1} \rangle}_{\text{trivial representation}} \oplus \underbrace{\{x \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}}_{\text{standard representation}}.$$

- The Schreier graph on $[n]$ arises from the permutation action of S_n on $\mathbb{R}^n$.
- The trivial summand gives the top eigenvector $\mathbf{1}$.
- The standard representation is the first nontrivial part of the point action.

Aldous' conjecture in representation-theoretic form

The second eigenvalue of the Cayley graph $\mathrm{Cay}(S_n, \Sigma)$ is already attained on the standard representation.

Progress on the original conjecture

Some partial progress

- Uniform distribution (Flatto-Odlyzko-Wales 1985).
- Coxeter case (Bacher 1994).
- Given rate case (Handjani-Jungreis 1996).
- Ferromagnetic XXZ-chain (Koma-Nachtergaele 1997).
- Multipartite set of transpositions (cesi 2010).
- Large hypercubes (Starr-Conomos 2011)

Theorem 1 (Caputo–Liggett–Richthammer 2010 JAMS)

For every connected weighted graph, the random walk and the interchange process have the same spectral gap; equivalently, they have the same relaxation time.

The generalised Aldous problem

Aldous property

For a symmetric generating set $\Sigma \subseteq S_n$, we say that $\mathrm{Cay}(S_n, \Sigma)$ has the *Aldous property* if its second eigenvalue is attained by the standard representation; equivalently,

$$\lambda_2\big(\mathrm{Cay}(S_n, \Sigma)\big) = \lambda_2\big(\mathrm{Sch}(S_n, [n], \Sigma)\big).$$

Main question

Which symmetric generating sets $\Sigma \subseteq S_n$ give a Cayley graph $\mathrm{Cay}(S_n, \Sigma)$ with the Aldous property?

Huang-Huang-Cioabă's conjecture 2017

Full-flag Johnson graphs possess the Aldous property.

Full-flag Johnson Graphs

arXiv:2603.10406.



Based on a joint work with Gary Greaves

A full flag arising from a permutation

Definition

A *full flag* of $[n]$ is a chain

$$U_1 \subset U_2 \subset \cdots \subset U_n = [n], \qquad |U_i| = i.$$

If $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$, define

$$U_i = \{\sigma_1, \ldots, \sigma_i\} \qquad (1 \leqslant i \leqslant n).$$

Then σ determines a full flag, and conversely every full flag arises in this way.

$$3142 \iff \{3\} \subset \{1,3\} \subset \{1,3,4\} \subset [4]$$

In this way, one may think of a full flag simply as a permutation together with all of its initial segments.

The full-flag Johnson graph $FJ(n, k)$

Definition

The vertices of $FJ(n, k)$ are all full flags of $[n]$. Two vertices are adjacent if the corresponding flags differ at exactly k levels.

Example 2

$FJ(n, 1)$ is the permutahedron.

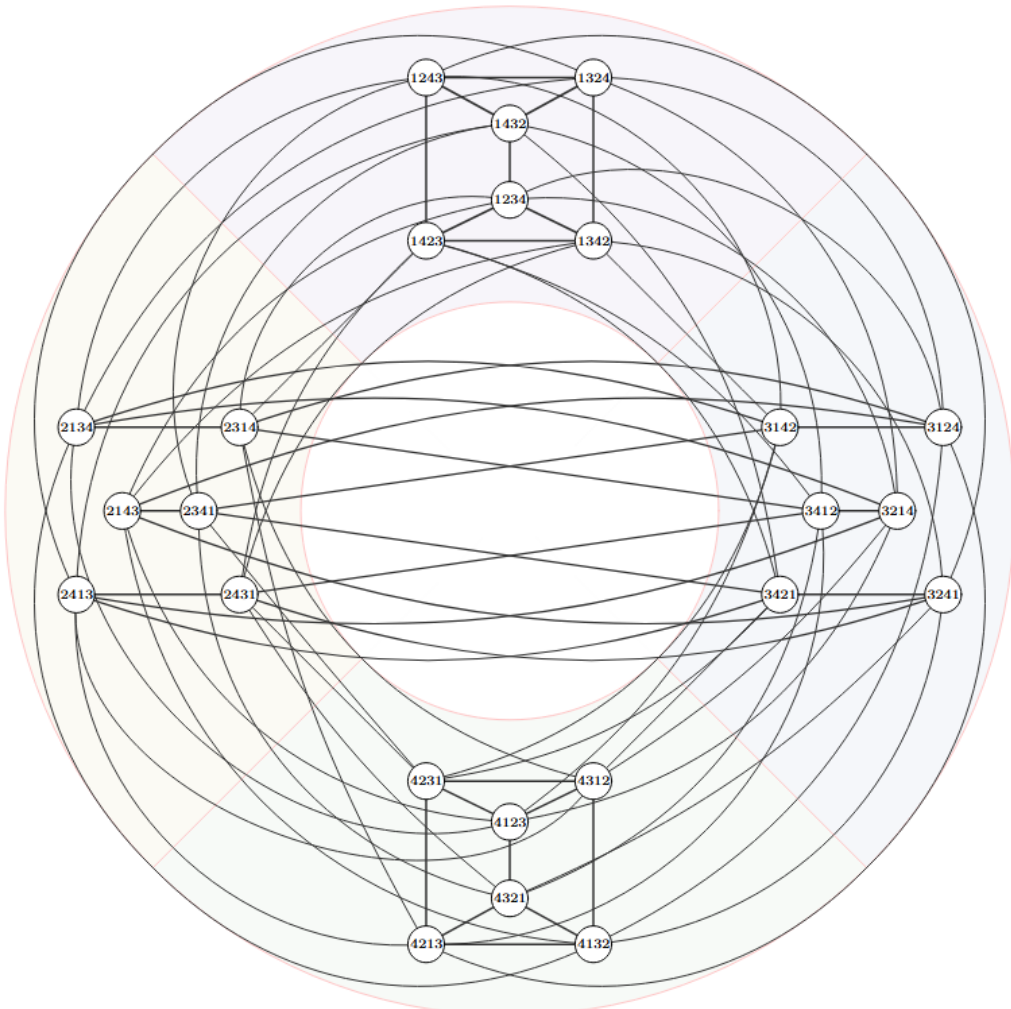
Example 3

1234	1342
$\{1\} \subset \{1, 2\} \subset \{1, 2, 3\} \subset [4]$	$\{1\} \subset \{1, 3\} \subset \{1, 3, 4\} \subset [4]$

These two vertices are adjacent in $FJ(4, 2)$: they agree at levels 1 and 4, and differ at levels 2 and 3.

The point-stabiliser partition of $FJ(4,2)$

Each coloured sector consists of permutations sending 1 to a fixed point.



Main result

Huang-Huang-Cioabă's conjecture 2017

Full-flag Johnson graphs $FJ(n, 2)$ possess the Aldous property.

We confirm this conjecture, that is,

Theorem 4 (Greaves–Zhu 2026+)

For every $n \geqslant 4$, the full-flag Johnson graph $FJ(n, 2)$ has the Aldous property.

Thank you for your attention!

Further reading:

G. Greaves and H. Zhu, *Aldous property for full-flag Johnson graphs*, preprint, arXiv:2603.10406, submitted.

G. Greaves and H. Zhu, *Aldous order and dominance order in the interchange process*, to appear very soon.