Eigenvalue multiplicity of the wreath product Cayley graphs

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Cayley graphs

Let G be a finite group and $\mathcal{S} = \mathcal{S}^{-1} \subseteq G \setminus \{e\}$. Vertices are group elements; an edge is one allowed move.

$$\Gamma = \mathrm{Cay}(G, \mathcal{S}), \qquad \underbrace{g}_{\text{current state}} \xrightarrow{s} \underbrace{gs}_{\text{after one move}} \qquad (s \in \mathcal{S}).$$

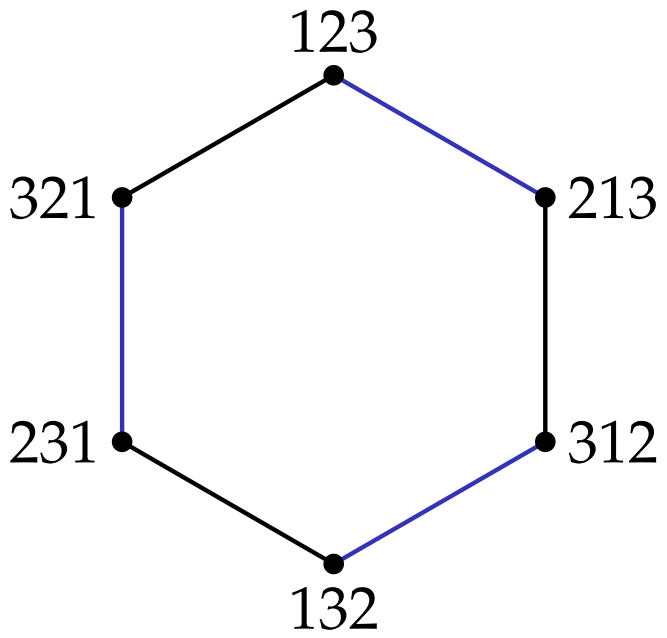
Example: $G = S_3$

$$\mathcal{S} = \{r_2, r_3\} = \{(12), (13)\}.$$

Vertices: the six permutations.

r_2: reverse the first two entries.

r_3: reverse all three entries.



Pancake graphs

Take $G = S_n$. The prefix reversal r_k reverses the first k positions, for $2 \leqslant k \leqslant n$. For example,

$$(3,1,4,2) \xrightarrow{r_3} (4,1,3,2).$$

Each flip is its own inverse: $r_k^2 = e$.

These generators define the pancake graph

$$P_n = \mathrm{Cay}(S_n, \{r_2, \ldots, r_n\}).$$

It has $n!$ vertices and $n-1$ neighbours at each vertex.

For $n = 4$: 24 vertices and three flips, r_2, r_3, r_4.

From flips to integer eigenvalues

Dalfó–Fiol (2020) obtain integer eigenvalues of P_n by summing the permutation matrices of the flips.

Each flip also permutes coordinates. For $n = 4$:

A single flip	Flip matrix	Sum
	$R(r_3)$	$M_4 = \sum_{k=2}^{4} R(r_k)$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \xrightarrow{r_3} \begin{pmatrix} x_3 \\ x_2 \\ x_1 \\ x_4 \end{pmatrix} \quad = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

$$\mathrm{Spec}(M_4) = \{-1, 0, 2, 3\} \subseteq \mathrm{Spec}(P_4).$$

For general n, write $M_n = \sum_{k=2}^{n} R(r_k)$.

The standard representation of S_n

For $n = 4$, compare two eigenvectors of M_4:

Constant vector $\mathbf{1}$

$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \xrightarrow{M_4} \begin{pmatrix}3\\3\\3\\3\end{pmatrix}$$

$\lambda = 3$

Sum-zero vector v

$$\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} \xrightarrow{M_4} \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$\lambda = 0$

$$\mathbb{C}^n = \mathbb{C}\mathbf{1} \oplus W_n, \qquad W_n = \{v : \textstyle\sum_i v_i = 0\}.$$

Permuting coordinates on W_n gives the standard representation.

Dalfó–Fiol (2020), for $n \geqslant 2$:

$$\mathrm{Spec}(M_n) = \{n-1\} \cup \{-1, 0, \ldots, n-2\} \setminus \{\lfloor n/2 \rfloor - 1\}.$$

Constants give $n-1$; sum-zero vectors give the other eigenvalues.

M_n is a quotient matrix of P_n, tracking the position of 1.

Signed and coloured permutations

A signed flip ($m = 2$)

$$(-3,1,2) \xrightarrow{\text{reverse first two}} (1,-3,2) \xrightarrow{\text{change signs}} (-1,3,2).$$

A coloured flip ($m = 3$): superscripts record colours.

$$(3^1,1^0,2^0) \xrightarrow{\text{reverse first two}} (1^0,3^1,2^0) \xrightarrow{\text{add } 1} (1^1,3^2,2^0).$$

In general, $r_k^{\pm}$ reverses the first k entries and adds ± 1 to their colours modulo m.

The coloured permutations form the wreath product $\mathbb{Z}_m \wr S_n$.

$$\mathcal{P}_m(n) = \mathrm{Cay}\big(\mathbb{Z}_m \wr S_n, \{r_k^{\pm} : 1 \leqslant k \leqslant n\}\big) \quad (m,n \geqslant 2).$$

$m = 2$: the burnt pancake graph on B_n.

non-normal

Groups behind wreath product Cayley graphs

The generalised symmetric group is

$$S(m,n) = \mathbb{Z}_m \wr S_n, \qquad B_n = S(2,n).$$

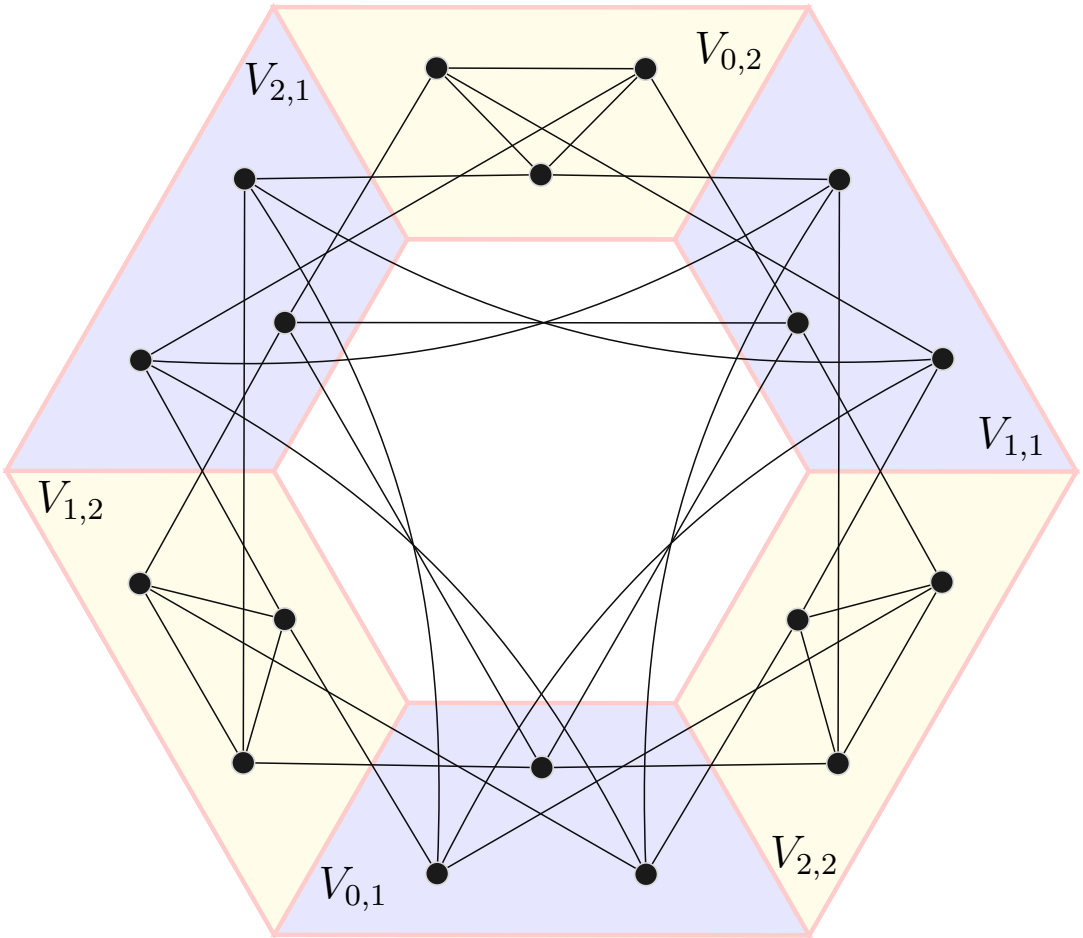
Group	Letters	Permutation matrix
S_n	n ordinary letters	$n \times n$
B_n	$2n$ signed letters	$2n \times 2n$
$S(m,n)$	mn coloured letters	$mn \times mn$

Forgetting colours recovers the ordinary permutation:

$$(3^1, 1^0, 2^0) \longmapsto (3,1,2) \in S_3.$$

The underlying S_n action on the sum-zero space W_n is the standard representation.

A generalised pancake graph



$\mathcal{P}_3(2)$

18 vertices
6 cells

An equitable partition

Each cell fixes the colour and position of the letter 1.

Vertices in a cell have equal neighbour counts into every cell.

Integer eigenvalues and multiplicities

Blanco and Buehrle (2024, 2025) use Dalfó–Fiol's method with the actions on signed and coloured letters.

Write $T_n = M_n + I$, including the ordinary identity flip $r_1 = e$.

$$\text{Burnt (2024):} \quad \operatorname{Spec}(T_n) \subseteq \operatorname{Spec}(\mathcal{P}_2(n)),$$

$$\text{Coloured (2025):} \quad 2\operatorname{Spec}(T_n) \subseteq \operatorname{Spec}(\mathcal{P}_m(n)) \quad (m \geqslant 3).$$

Blanco–Buehrle's multiplicity conjecture (2025)

For $4 \mid m$, $n \geqslant 2$ and $1 \leqslant k \leqslant n-1$,

$$\operatorname{mult}_{\mathcal{P}_m(n)}(2k) \geqslant \begin{cases} 3, & k \neq \lfloor n/2 \rfloor, \\ 2, & k = \lfloor n/2 \rfloor. \end{cases}$$

Order of the vertices

For $\mathcal{P}_3(2)$, label each quotient vertex by (c, j):
c is the colour and j the position of the letter 1.

Position first

$(0,1),(1,1),(2,1),(0,2),(1,2),(2,2)$

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 & 0 & 2 \end{array}\right)$$

Colour first

$(0,1),(0,2),(1,1),(1,2),(2,1),(2,2)$

$$\left(\begin{array}{cc|cc|cc} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 2 & 1 & 0 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 2 \end{array}\right)$$

Reordering the vertices reveals the block-circulant structure.

The multiplicity theorem

Block-circulant formula

$$D_n = \begin{pmatrix} 0 & & & 0 \\ & 1 & & \\ & & \ddots & \\ & & & n-2 \\ 0 & & & n-1 \end{pmatrix}, \quad E_n = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad T_n = D_n + E_n.$$

$$Q_{m,n} = \operatorname{circ}(2D_n, E_n, 0, \ldots, 0, E_n) \sim \bigoplus_{j=0}^{m-1} \left(2D_n + 2\cos\tfrac{2\pi j}{m} E_n\right).$$

Theorem (Greaves–Zhu, 2026)

Let $4 \mid m$ and $n \geqslant 2$. Then, for $1 \leqslant k \leqslant n-1$,

$$\operatorname{mult}_{\mathcal{P}_m(n)}(2k) \geqslant \begin{cases} 3, & k \neq \lfloor n/2 \rfloor, \\ 2, & k = \lfloor n/2 \rfloor. \end{cases}$$

Open problems

Blanco–Buehrle conjecture

For every $n \geqslant 4$, the burnt pancake graph is conjectured to satisfy

$$\{-(n-1), -(n-2), \ldots, n\} \subseteq \operatorname{Spec}(\mathcal{P}_2(n)).$$

Sawada–Williams conjecture

For every $n \geqslant 4$, is the graph

$$\operatorname{Cay}\big(S_n, \{r_{n-2}, r_{n-1}, r_n\}\big)$$

Hamiltonian? Only the three longest flips are allowed.

Blanco–Buehrle (2024), Conjecture 8 (range adjusted).
Sawada–Williams (2016), Conjecture 4; Blanco et al. (2026), Section 4.2.

Thank you for your attention

Further reading:

G. Greaves and H. Zhu, *A note on some spectral properties of generalised pancake graphs*, Discrete Math. **349**(9) (2026), 115102.