Card shuffling with adjacent swaps and Fill's Gap Problem

Haoran Zhu

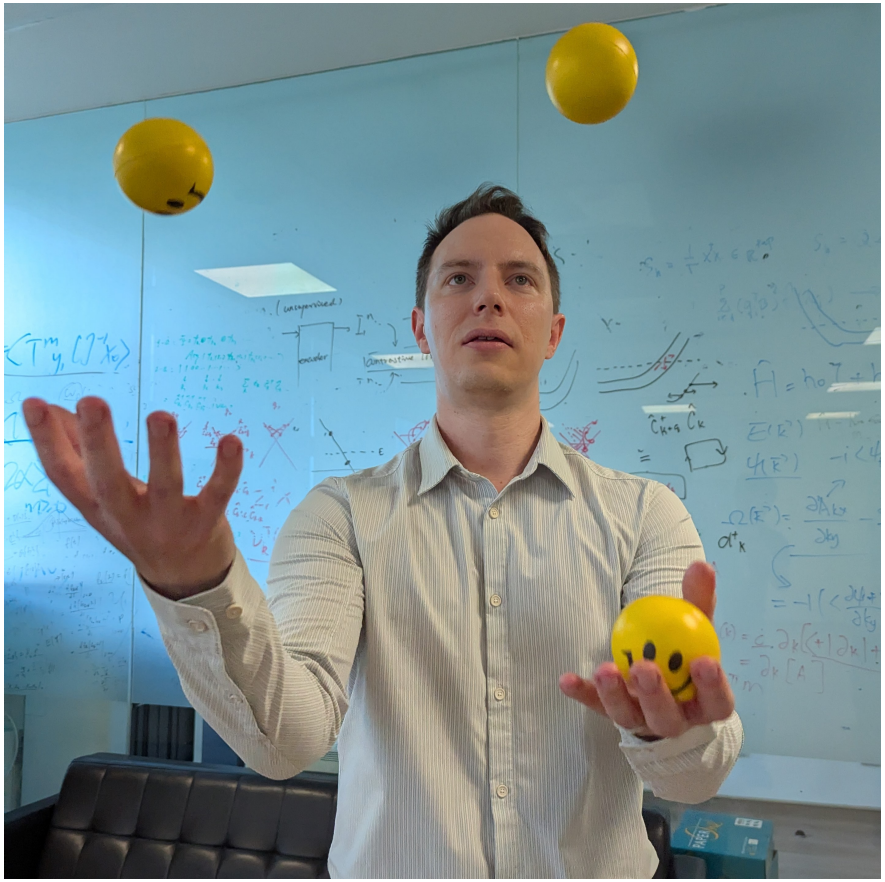
School of Physical and Mathematical Sciences,
Nanyang Technological University

5th November 2026

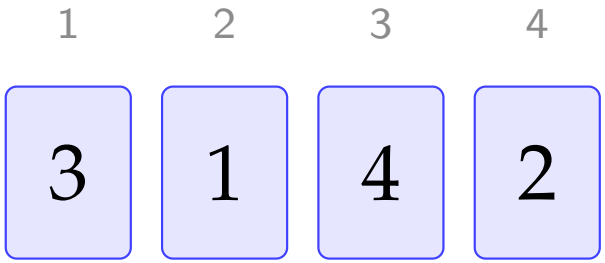
Plan

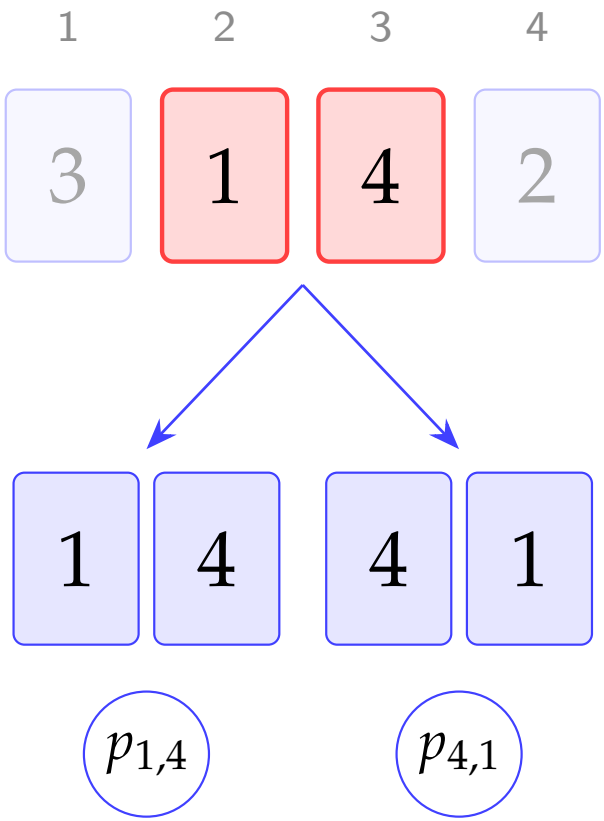
- Biased card shuffling and Fill's Gap Problem
- **Main result**: a sharp lower bound on the spectral gap
- Extremal chains and eigenvalue multiplicity

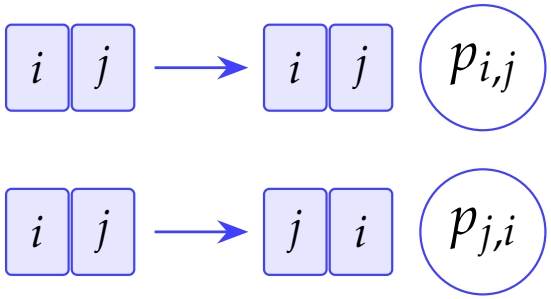
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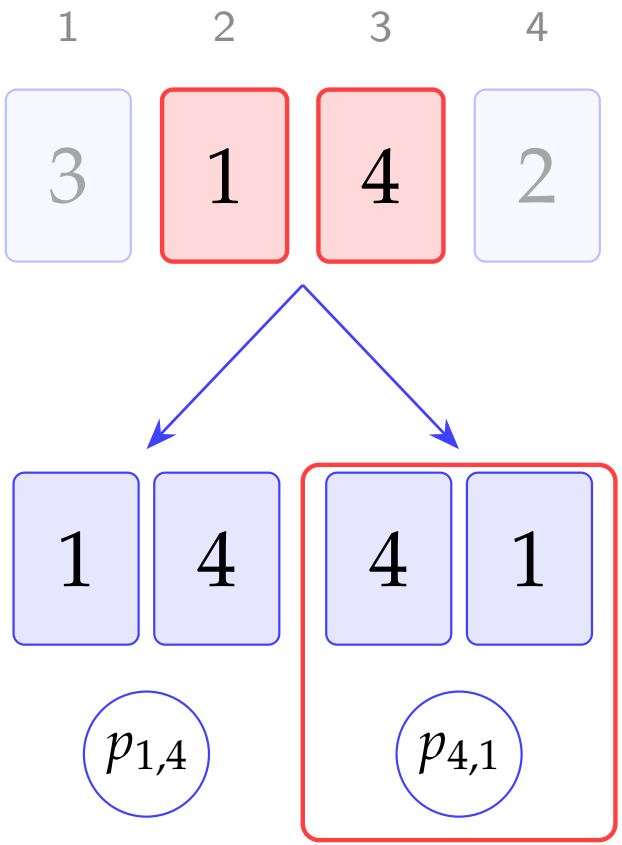


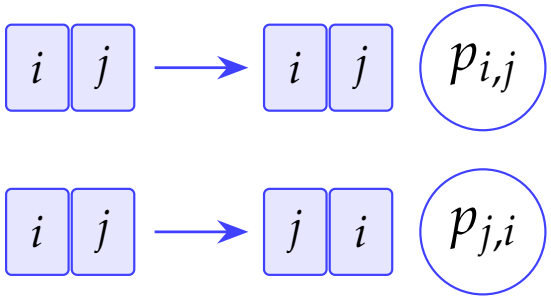
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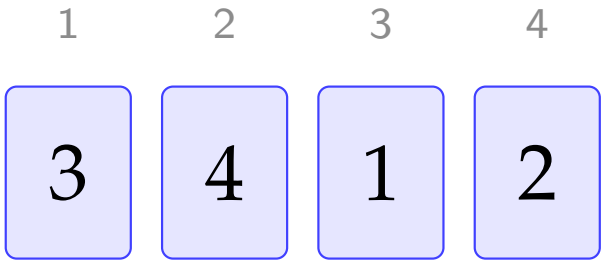

1
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3
1
4
2

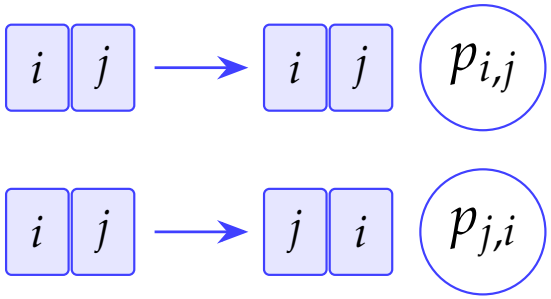

1
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1
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4
1
$p_{1,4}$
$p_{4,1}$

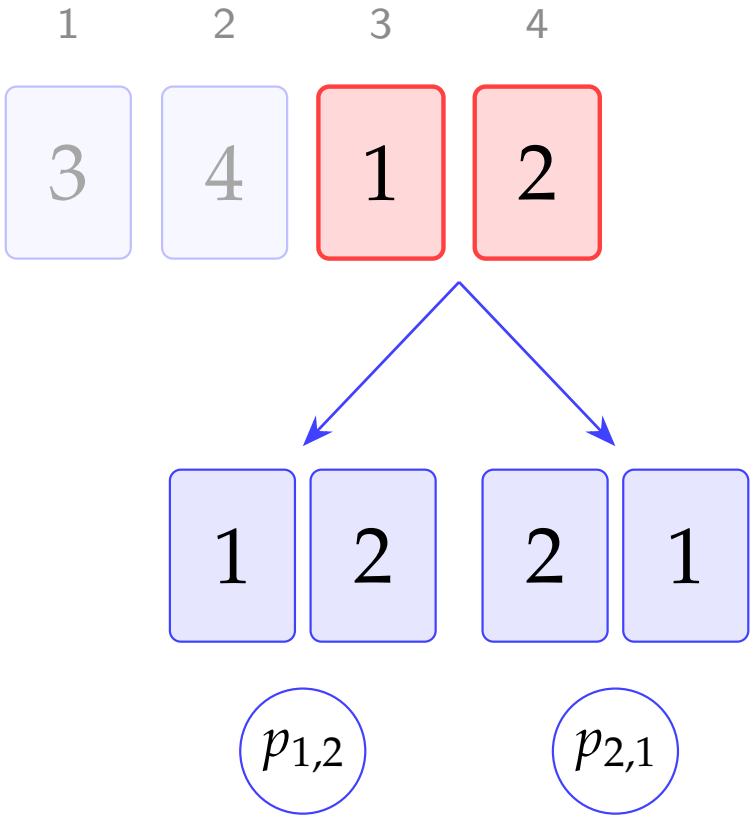

i
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i
j
$p_{i,j}$
i
j
j
i
$p_{j,i}$

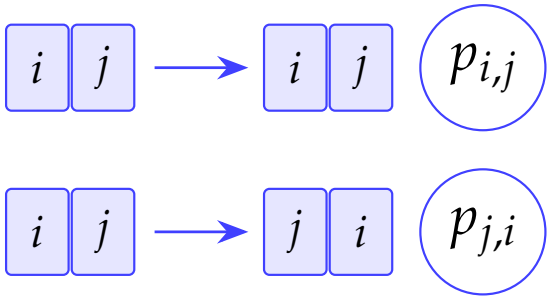

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4
1
$p_{1,4}$
$p_{4,1}$

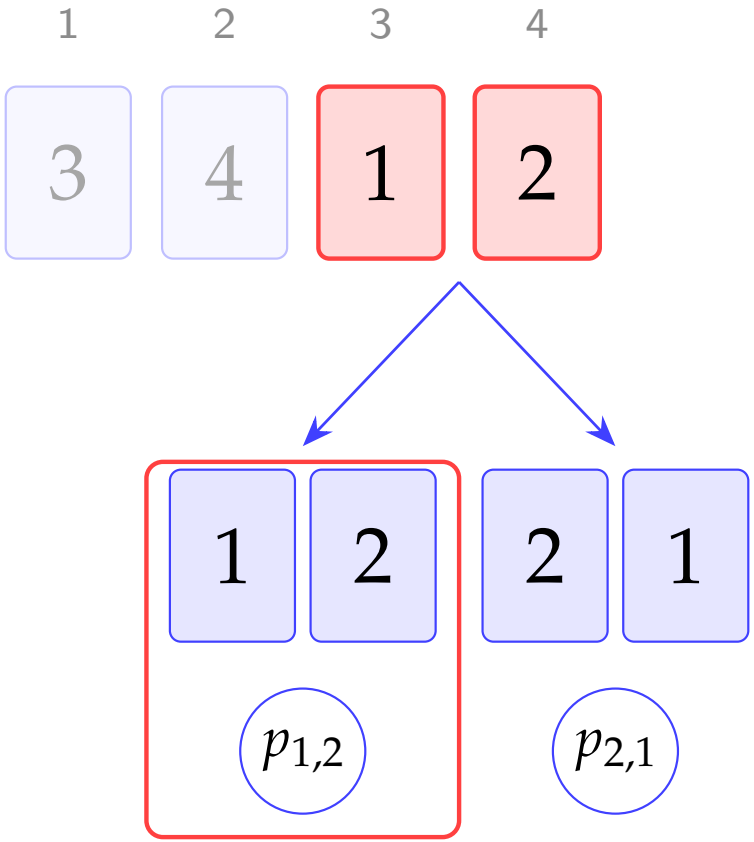

i j
i j
$p_{i,j}$
i j
j i
$p_{j,i}$

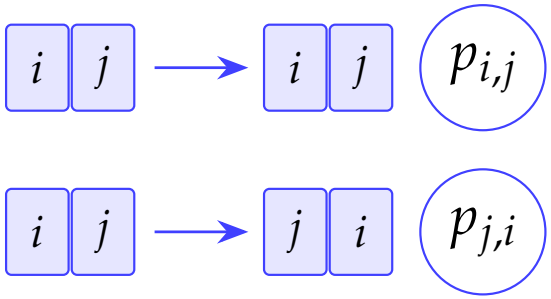

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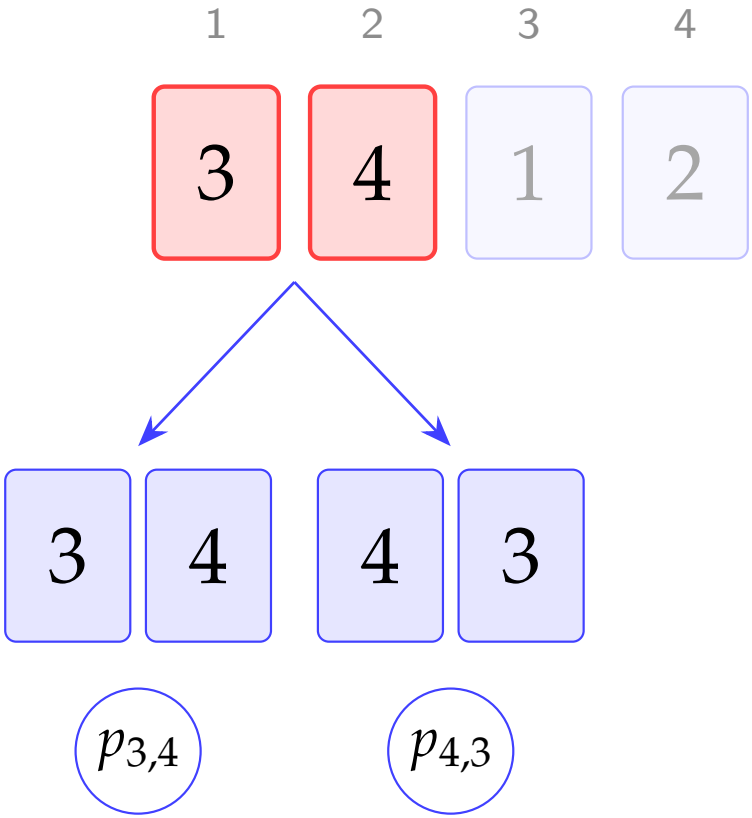

i
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$p_{i,j}$
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$p_{j,i}$

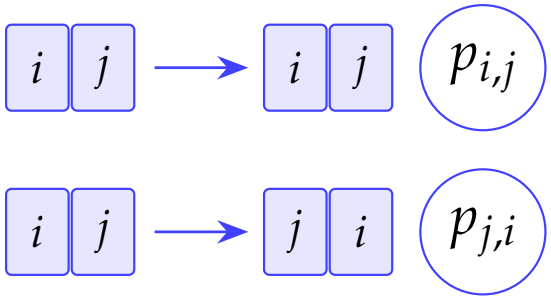

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$p_{1,2}$
$p_{2,1}$

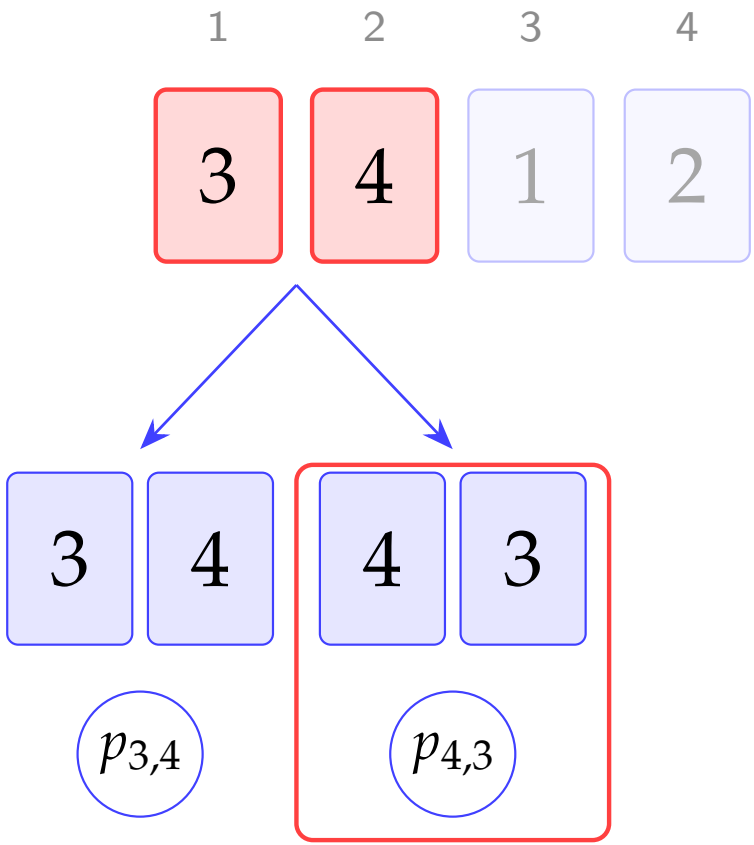

i j
i j
$p_{i,j}$
i j
j i
$p_{j,i}$

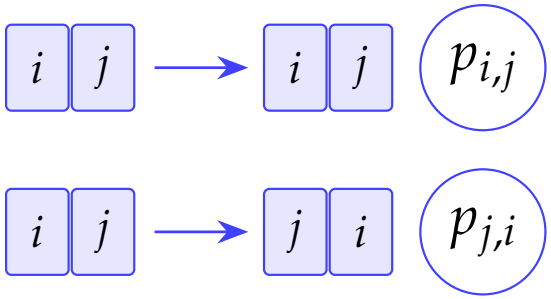

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$p_{1,2}$
$p_{2,1}$

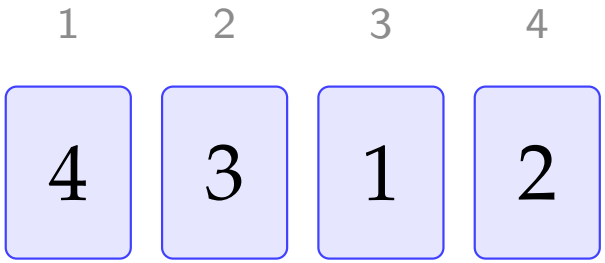

i j
i j
$p_{i,j}$
i j
j i
$p_{j,i}$

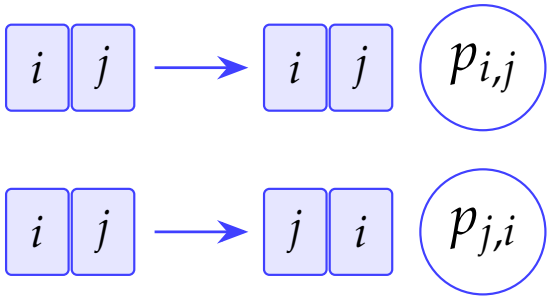

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3
$p_{3,4}$
$p_{4,3}$


i
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i
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$p_{i,j}$
i
j
j
i
$p_{j,i}$


1
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1
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$p_{3,4}$
$p_{4,3}$


i
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$p_{i,j}$
i
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i
$p_{j,i}$

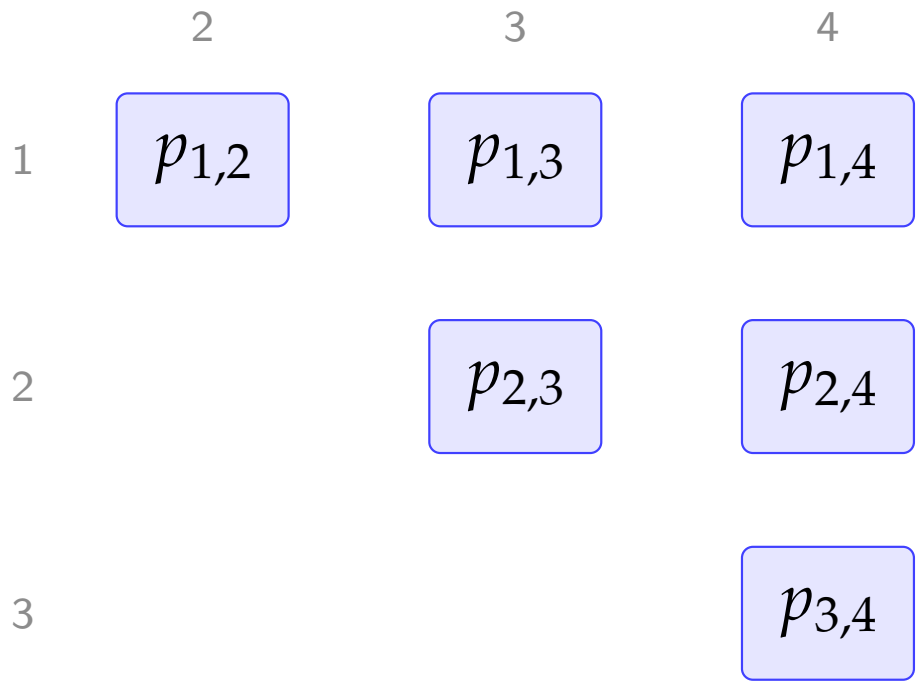


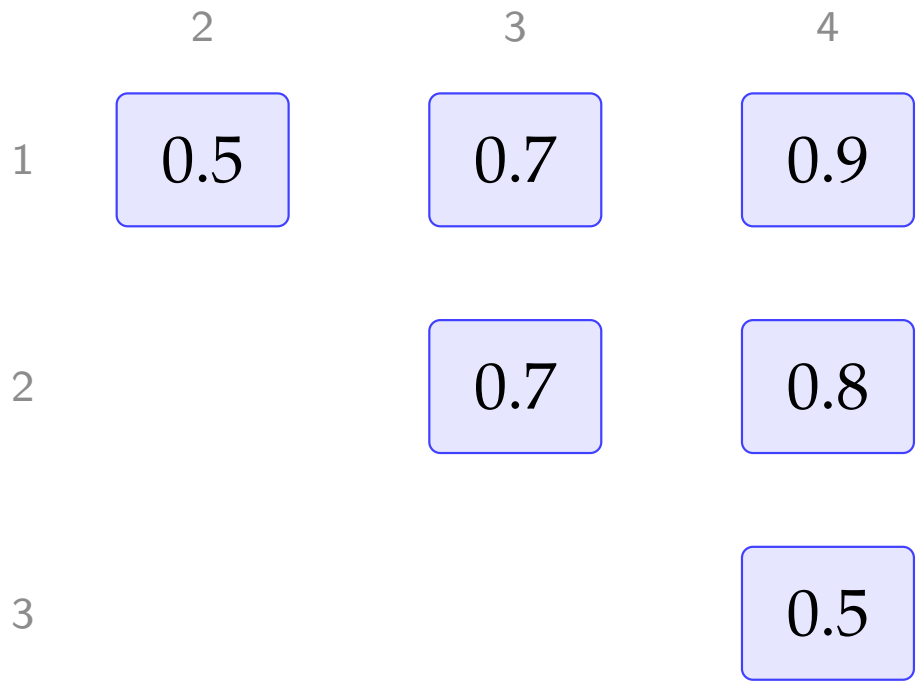


$$p_{i,j} + p_{j,i} = 1$$

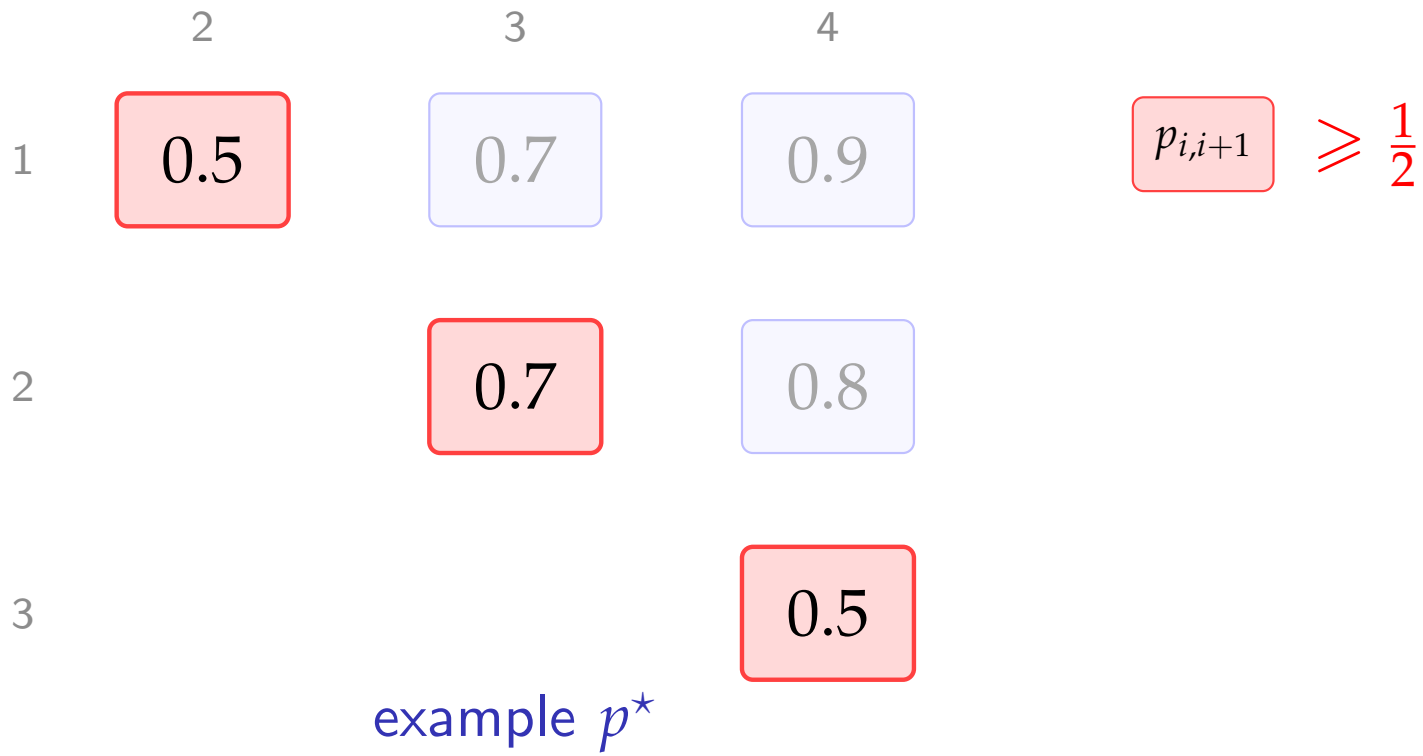
biased adjacent-transposition chain

on S_n

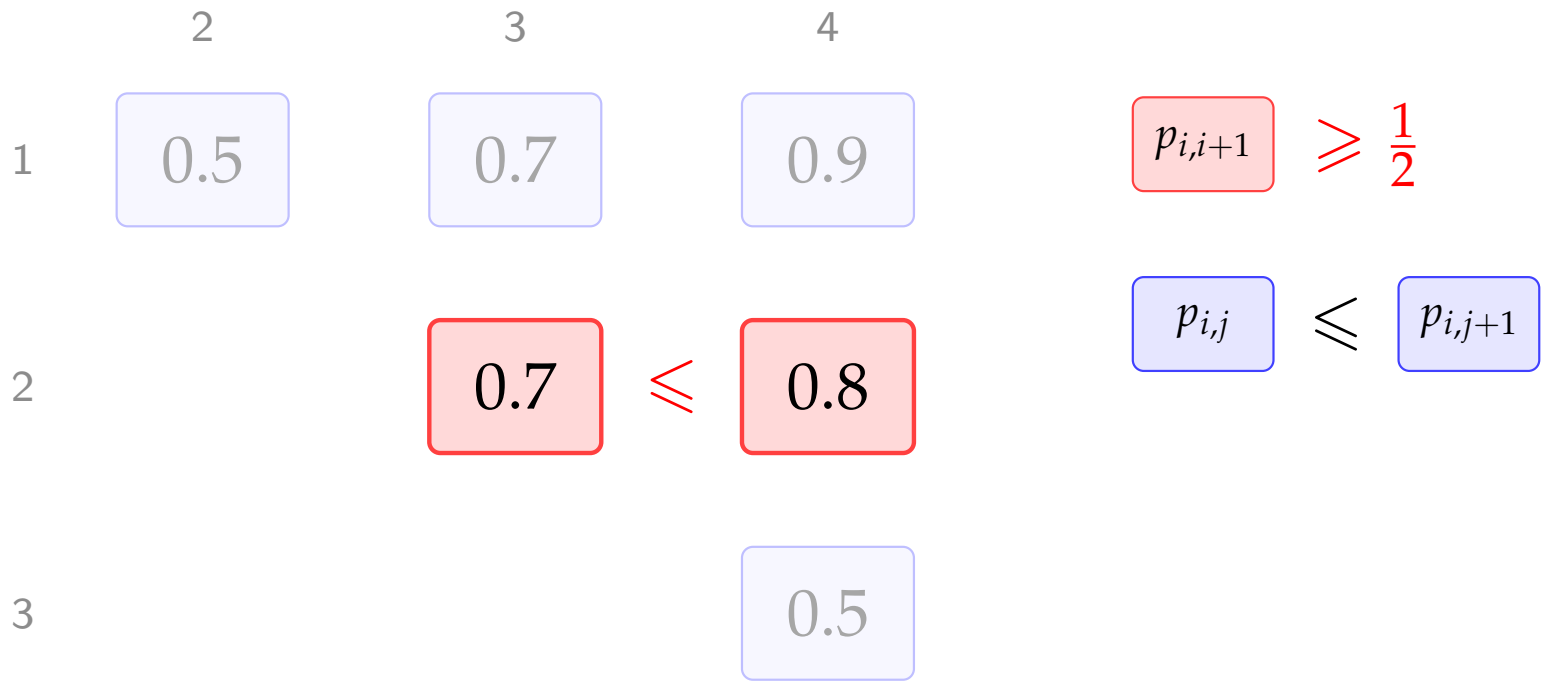

2
3
4
1
$p_{1,2}$
$p_{1,3}$
$p_{1,4}$
2
$p_{2,3}$
$p_{2,4}$
3
$p_{3,4}$

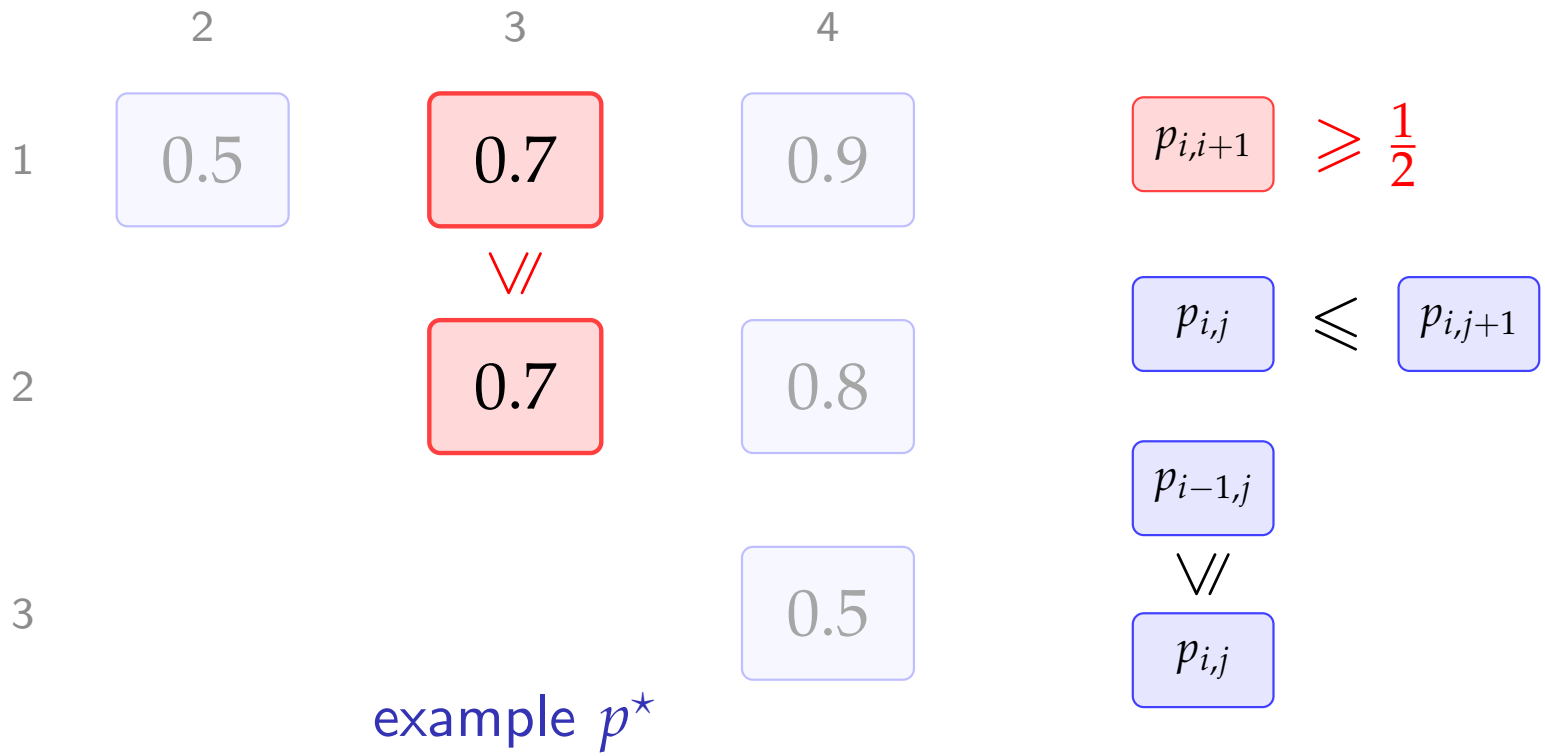


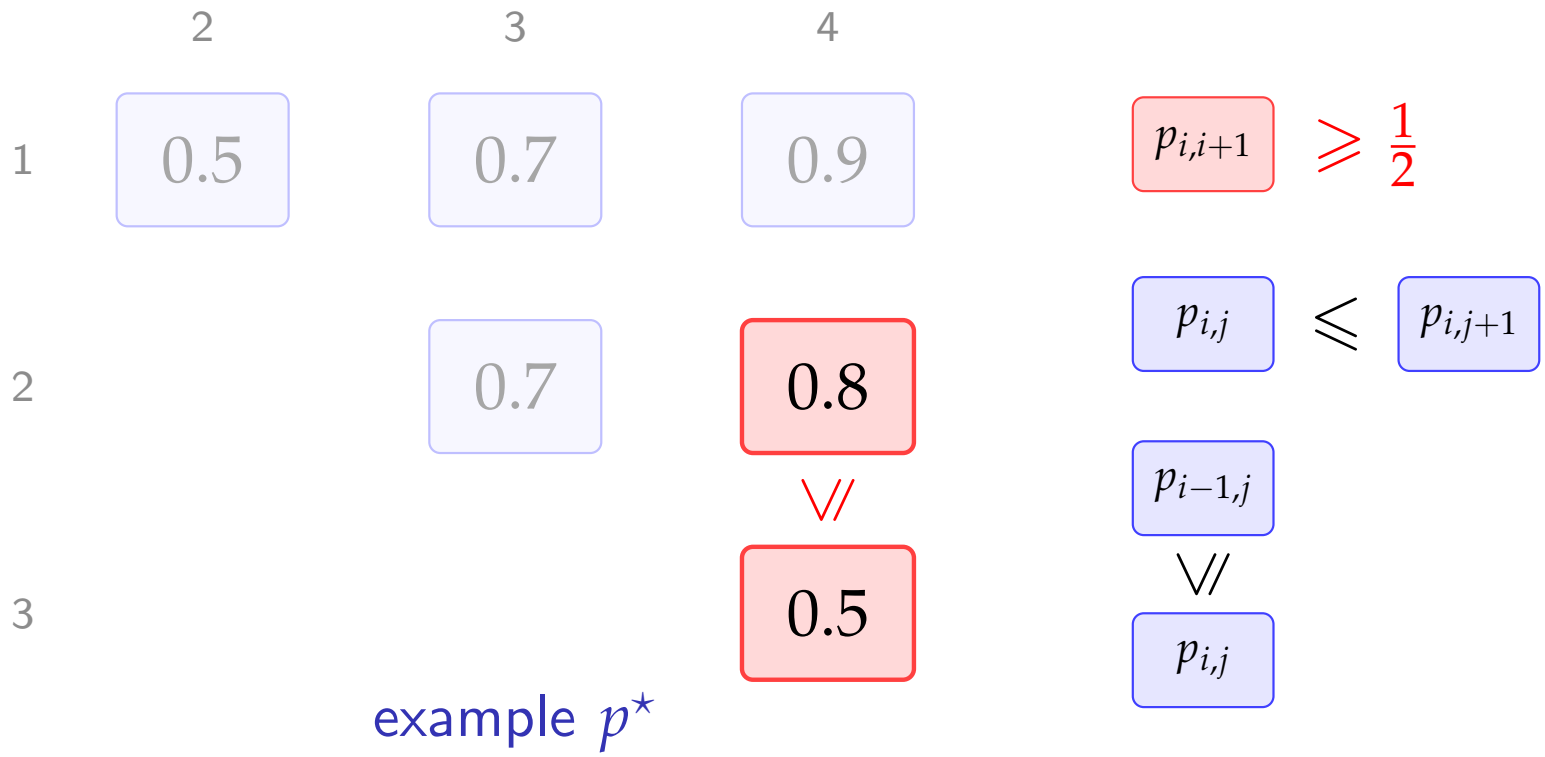
example $p^\star$

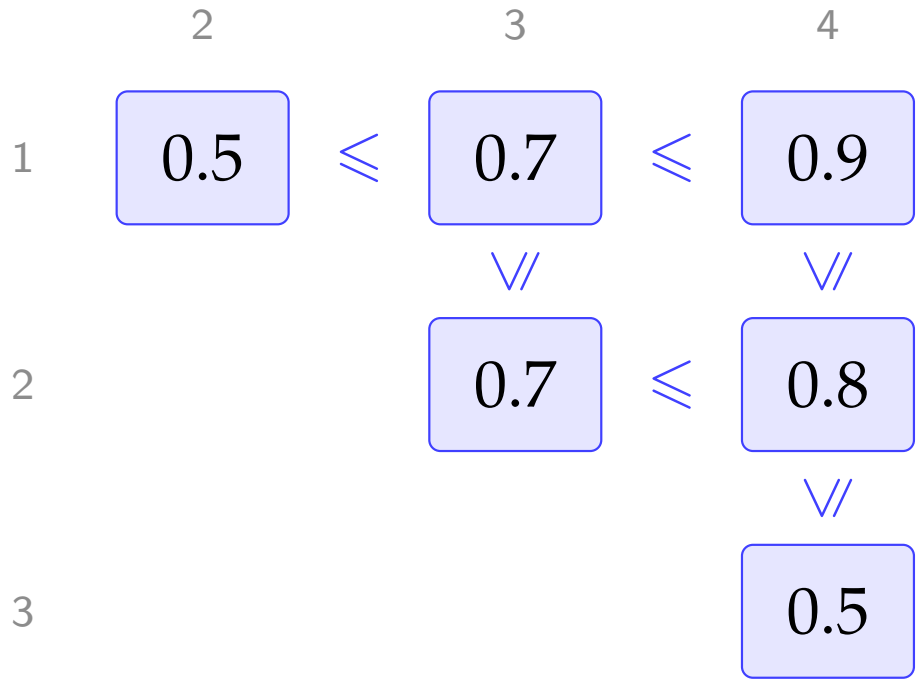

2
3
4
1
0.5
0.7
0.9
2
0.7
0.8
3
0.5
example $p^\star$
$p_{i,i+1} \geqslant \frac{1}{2}$


2
3
4
1
0.5
⩽
0.7
0.9
2
0.7
0.8
3
0.5
example $p^\star$
$p_{i,i+1} \geqslant \frac{1}{2}$
$p_{i,j} \leqslant p_{i,j+1}$

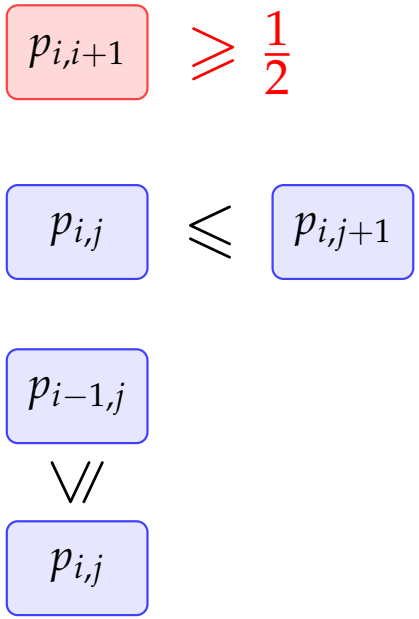

2
3
4
1
0.5
0.7
0.9
2
0.7
⩽
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3
0.5
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2
3
4
1
0.5
0.7
0.9
2
0.7
0.8
3
0.5
example $p^\star$
$p_{i,i+1} \geqslant \frac{1}{2}$
$p_{i,j} \leqslant p_{i,j+1}$
$p_{i-1,j}$
$p_{i,j}$


2
3
4
1
0.5
0.7
0.9
2
0.7
0.8
3
0.5
example $p^\star$
$p_{i,i+1} \geqslant \frac{1}{2}$
$p_{i,j} \leqslant p_{i,j+1}$
$p_{i-1,j}$
$\vee$
$p_{i,j}$



example $p^\star$



regular

Spectral gap

K: the $n! \times n!$ transition matrix of the chain (24×24 for $n = 4$).

Fill (2003): K is reversible with respect to $\mu(x) = Z^{-1} \prod_{u<v} p_{x_u, x_v}$, and K is similar to $I - K$.

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- Spectral gap $\lambda_K := 1 - \beta_K$, where β_K is the second-largest eigenvalue; it is the smallest positive eigenvalue of K.
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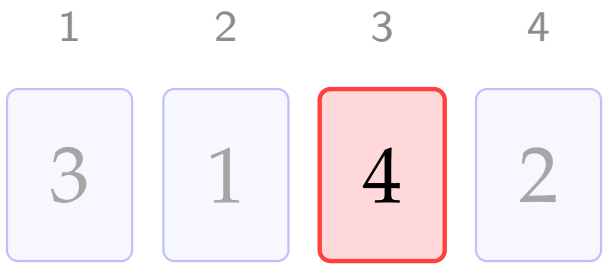
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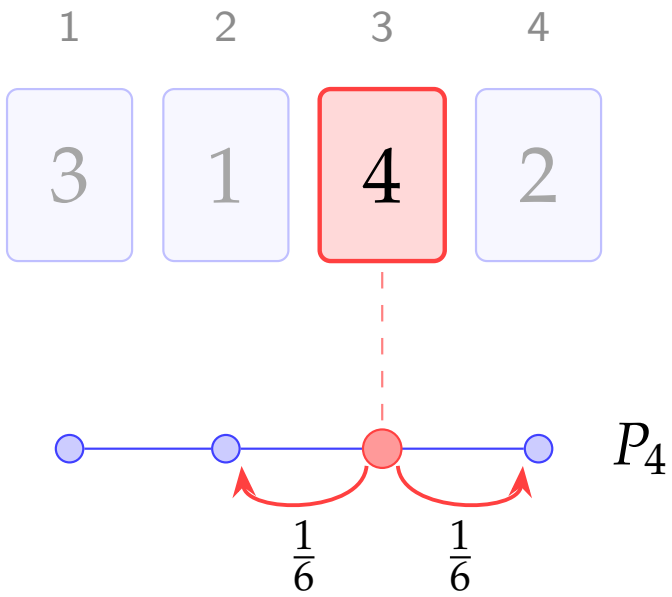
Example

$p^\star$: $\lambda_K = 0.1261$.

The unbiased chain $p \equiv \frac{1}{2}$

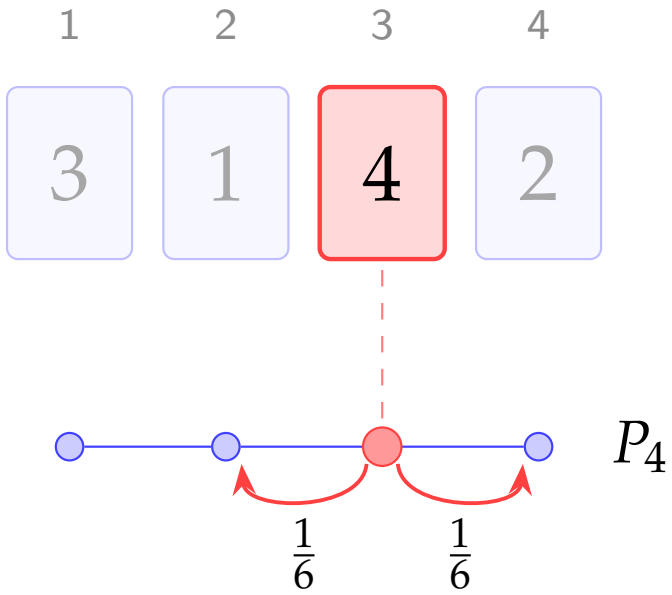


The unbiased chain $p \equiv \frac{1}{2}$



one card alone:
lazy random walk
on the path P_4

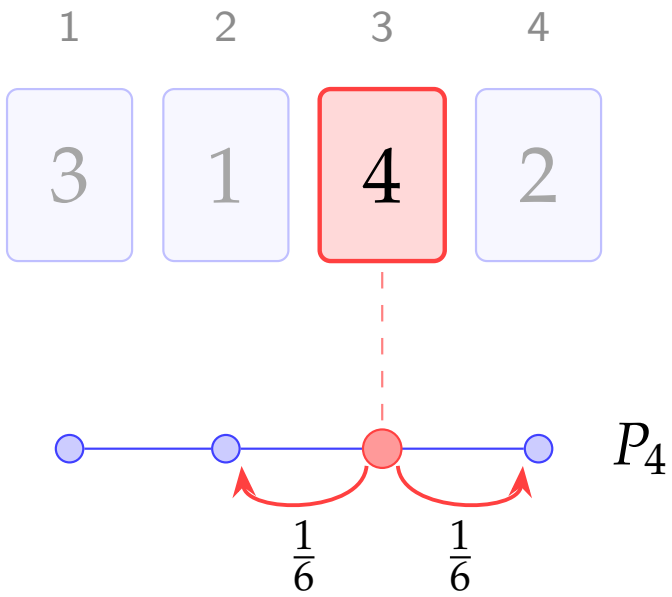
The unbiased chain $p \equiv \frac{1}{2}$



one card alone:
lazy random walk
on the path P_4

Wilson (2004): $\lambda_K = \dfrac{1-\cos(\pi/n)}{n-1} \sim \dfrac{\pi^2}{2n^3}$, the gap of this one-card walk.

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one card alone:
lazy random walk
on the path P_4

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Example

$n = 4$: $\lambda_K = \frac{1}{3}\left(1 - \cos\frac{\pi}{4}\right) = 0.0976 \ < 0.1261.$

Fill's Gap Problem

Gap Problem (2003). Prove or disprove: among all regular p, the spectral gap λ_K is minimised by the unbiased $p \equiv \frac{1}{2}$.

E.g. for $n = 4$: is $\lambda_K \geqslant 0.0976$ for every regular p?

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- Even *any* bound $1/\lambda_K \leqslant \mathrm{poly}(n)$ over all regular p was open.

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- Notes from 2003, posted to arXiv in 2025 [arXiv:2508.12557].

Two spectral gap problems

Aldous's conjecture; proved by Caputo, Liggett, Richthammer (2010): for *unbiased* swaps along the edges of any graph, the whole deck has the same gap as one card alone.

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- Aldous: one chain; compare the deck with one card.
- Gap Problem: many chains; compare different biases.
- Complementary. In the extremal case, again one card carries the gap.

Twenty years of special cases

2003	Fill	Gap Problem; $n = 3$	*notes*
2004	Wilson	$p \equiv \frac{1}{2}$	*AAP*
2005	Benjamini et al.	$p_{i,j} \equiv q$	*TAMS*

Twenty years of special cases

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2017	Haddadan, Winkler	3 classes	*STACS*
2018	Miracle, Streib	k classes	*LATIN*
2019	Haddadan, Winkler	3 classes	*TOCS*
2019	Labbé, Lacoin	cutoff, $p_{i,j} \equiv q$	*AoP*

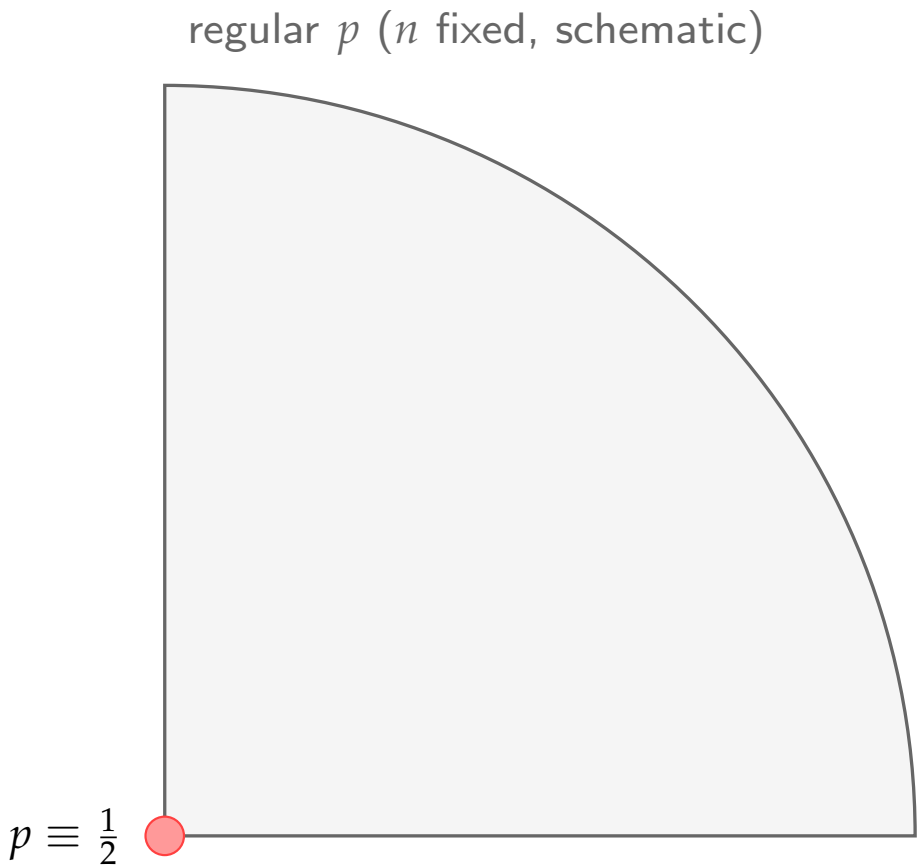
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2020	Greenberg, Randall, Streib	$p_{i,j} \equiv q$	*CPC*
2020	Miracle, Streib, Streib	k classes	*RANDOM*
2022	Bhakta et al.	$p_{i,j} = r_i$; trees	*RSA*
2024	Miracle, Streib	k classes	*SIDMA*
2025	Miracle, Streib, Streib	k classes	*ToC*

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2026	**Greaves, Zhu**	**all regular** p	

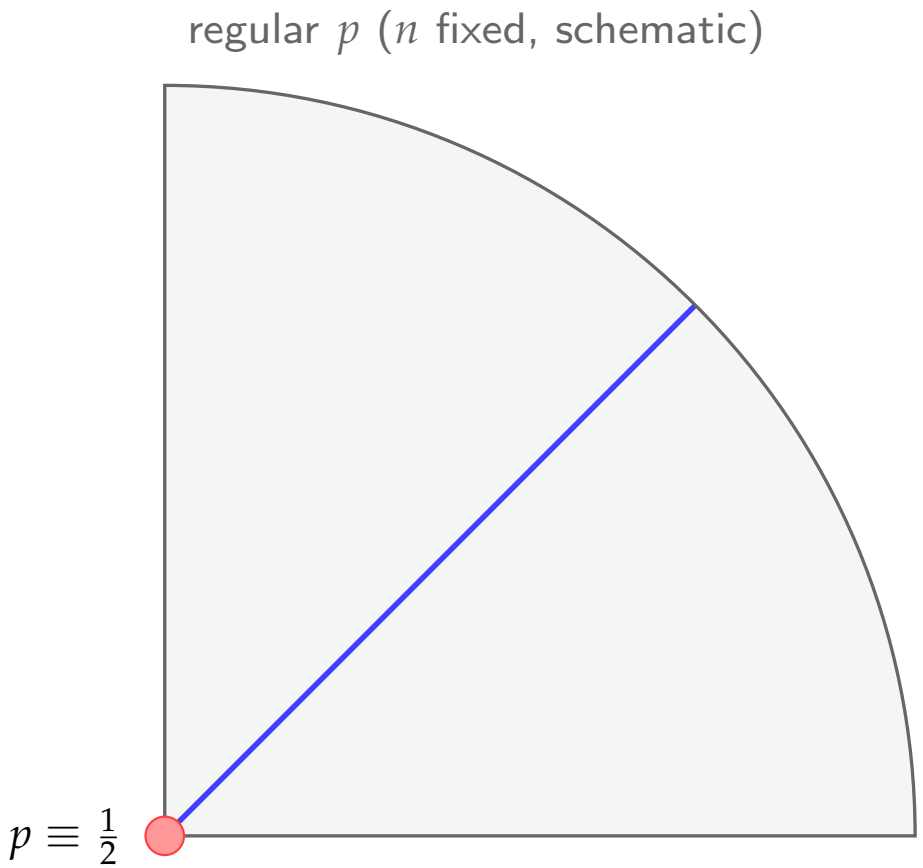
The space of regular p



point: $p \equiv \frac{1}{2}$

2004, 2016

The space of regular p



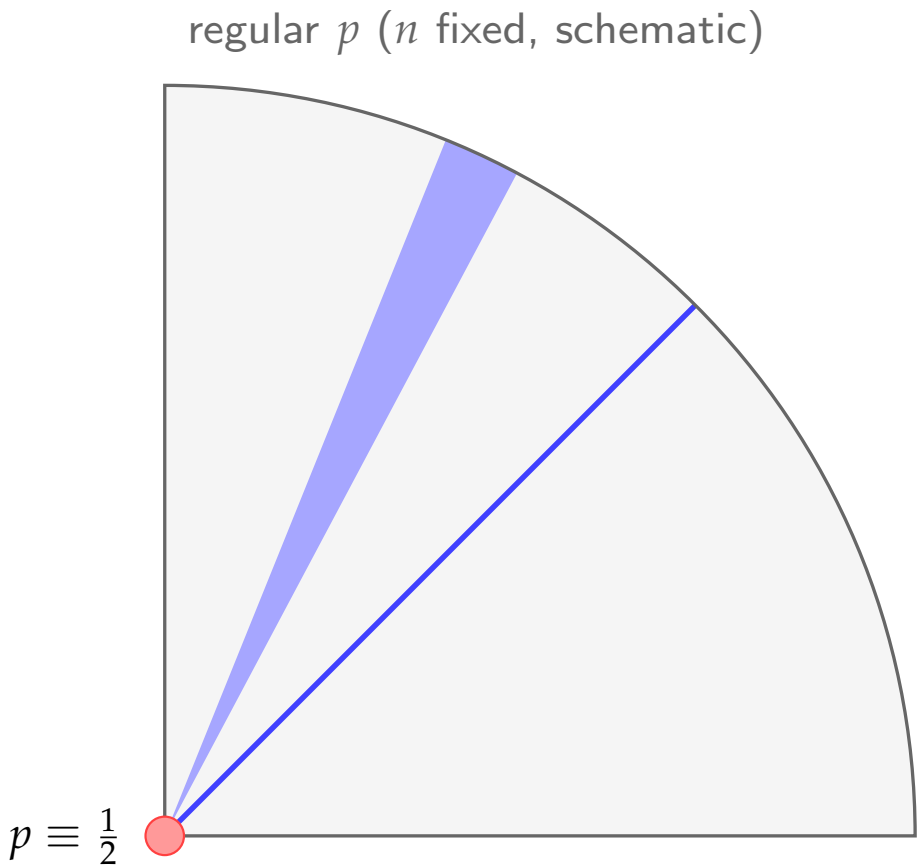
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The space of regular p

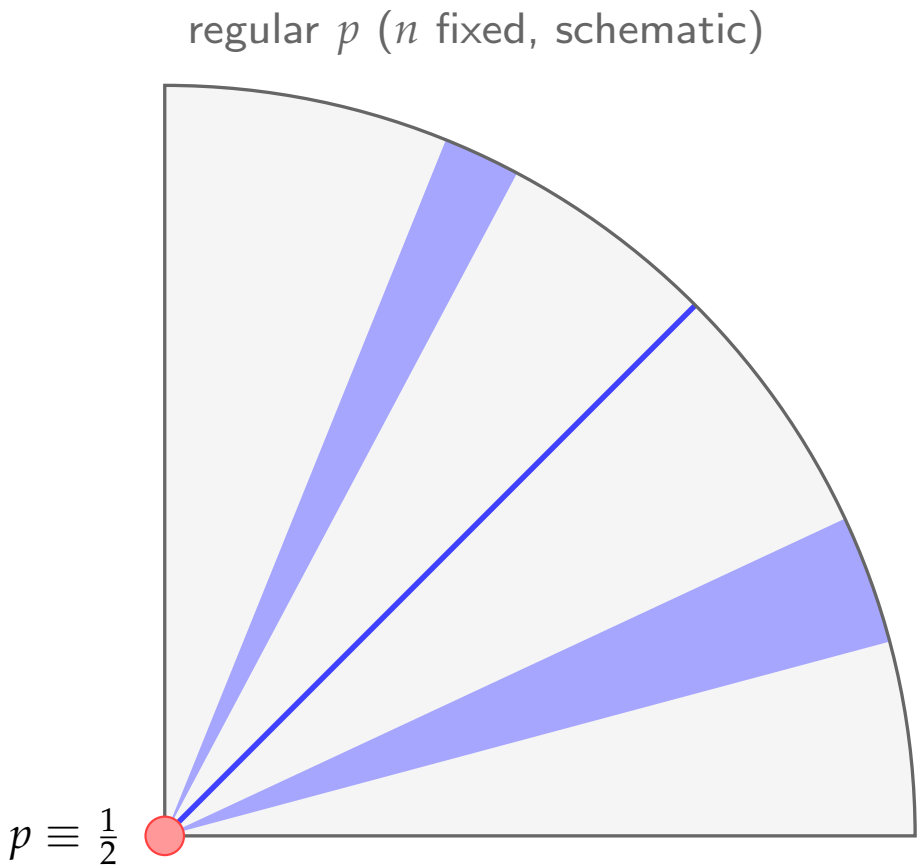


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The space of regular p



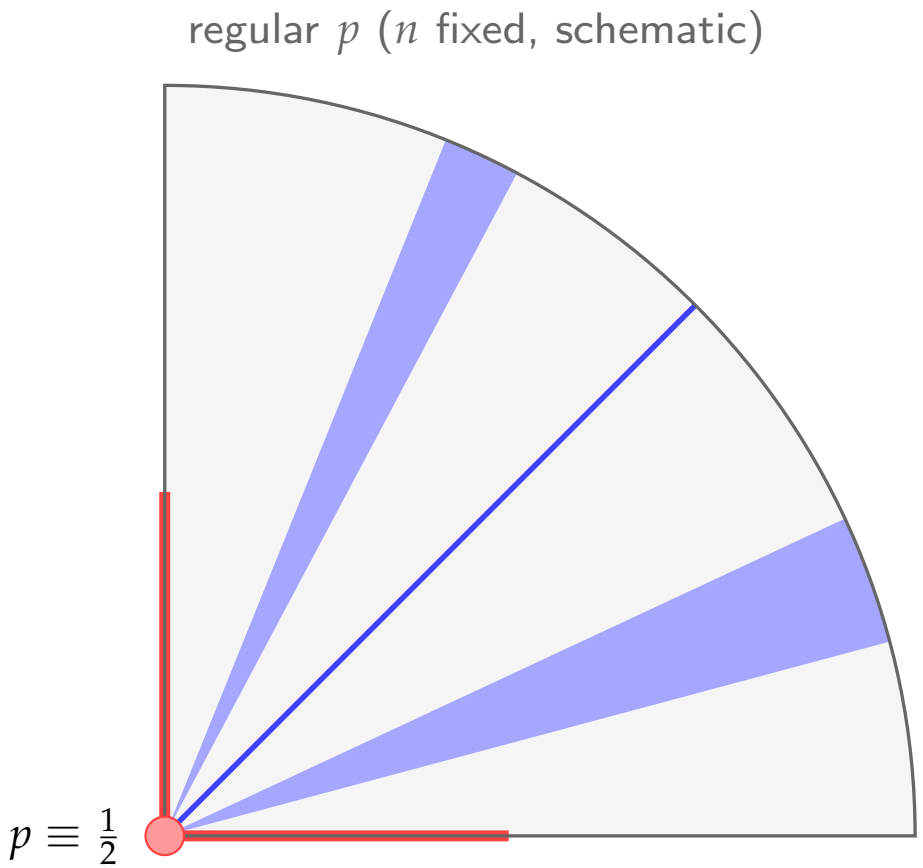
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The space of regular p



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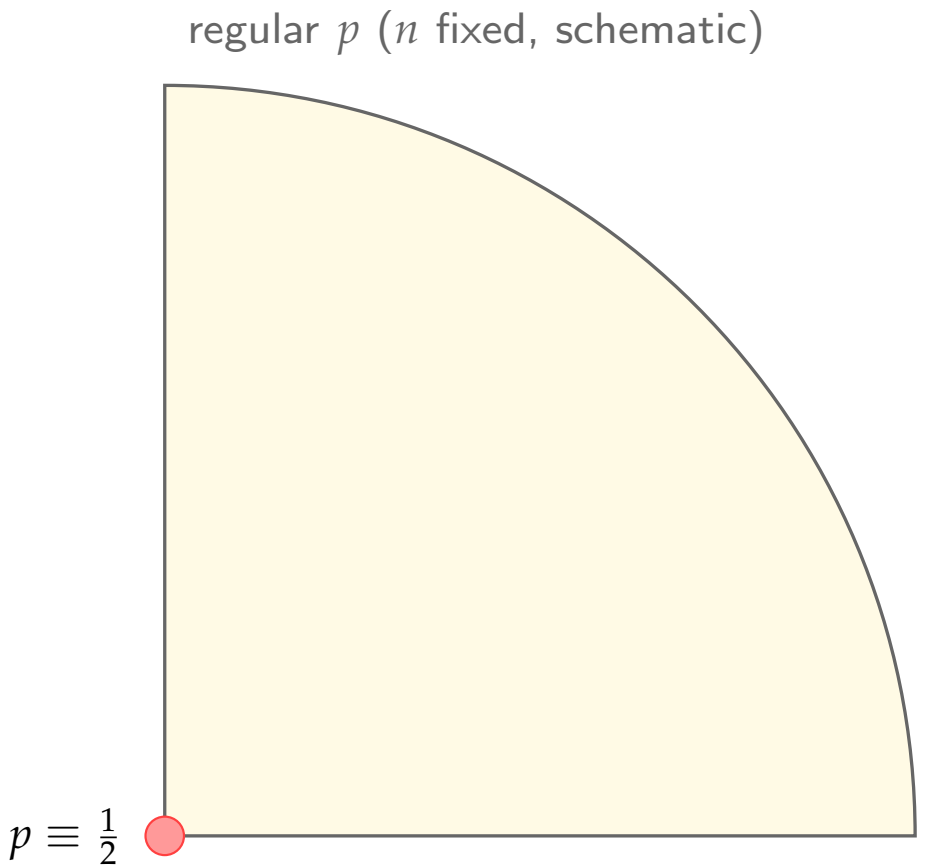
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where the gap is smallest

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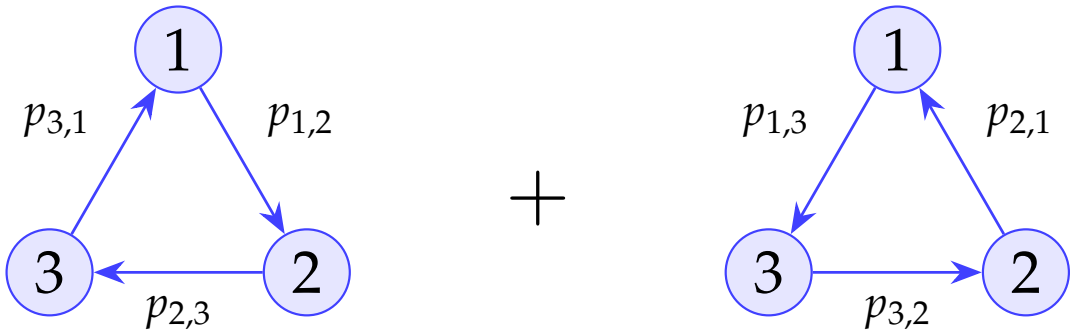
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all regular p
2026, Greaves and Zhu

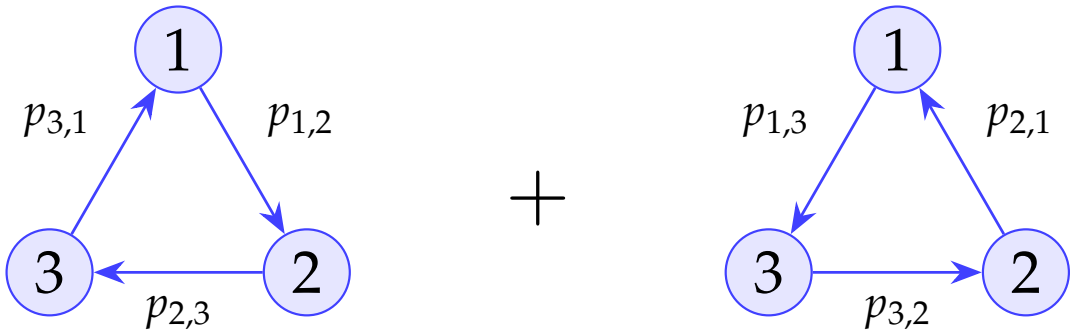
Three cards

Fill (2003): For $n = 3$, $\lambda_K = \frac{1}{2}(1 - m)$, where $m = \sqrt{p_{1,2}\, p_{2,3}\, p_{3,1} + p_{3,2}\, p_{2,1}\, p_{1,3}}$.



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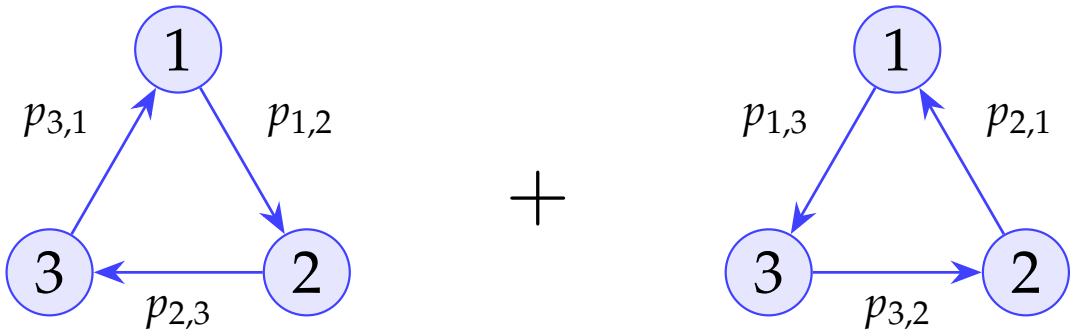
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- Read $p_{i,j}$ as "i beats j": m^2 is the chance that three independent games form a cycle.

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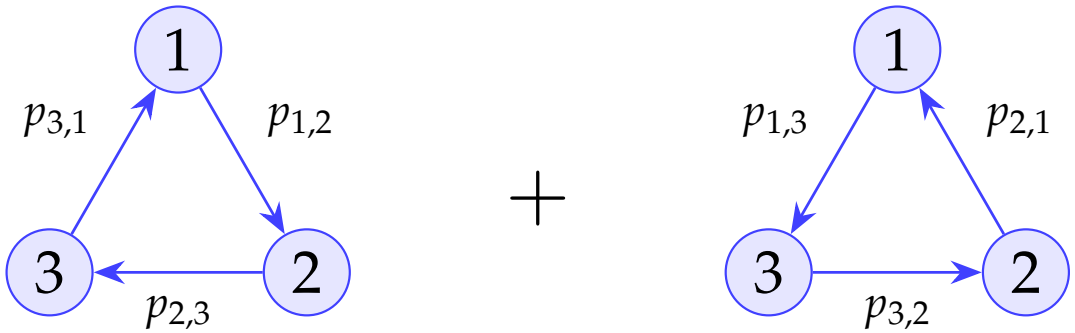
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- For general n: $m_p :=$ the largest m over all triples $i < j < k$.
- $p \equiv \frac{1}{2}$: $m_p = \frac{1}{2}$; p regular $\Rightarrow m_p \leqslant \frac{1}{2}$.

Main result

Greaves and Zhu (2026): For $n \geqslant 3$ and every p,

$$\lambda_K \geqslant \frac{1 - 2\, m_p \cos(\pi/n)}{n-1}.$$

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Fill's spectral gap conjecture is true: among all regular p, λ_K is minimised by $p \equiv \frac{1}{2}$. Moreover $1/\lambda_K \leqslant n^3/2$.

The example $p^\star$

Lower bound for λ_K at $p^\star$ $(n = 4)$.

	2	3	4
1	0.5	0.7	0.9
2		0.7	0.8
3			0.5

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- Unbiased value: $\lambda_K \geqslant \frac{1}{3}\left(1 - \cos\frac{\pi}{4}\right) = 0.0976$

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- True value: $\lambda_K = 0.1261$

More examples

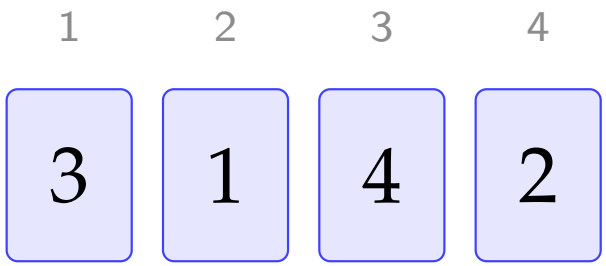
		weights $w_i = 2^{-i}$	
n	$p \equiv \frac{1}{2}$	bound	λ_K
3	0.2500	0.2892	0.2892
4	0.0976	0.1346	0.1401
5	0.0477	0.0794	0.0909
6	0.0268	0.0539	0.0678
7	0.0165	0.0400	0.0550
⋮	⋮	⋮	
large n	$\approx 4.93/n^3$	$\approx 0.157/n$	

$p_{i,j} = w_i/(w_i + w_j)$: the self-organising lists behind the Gap Problem.

Idea of the proof

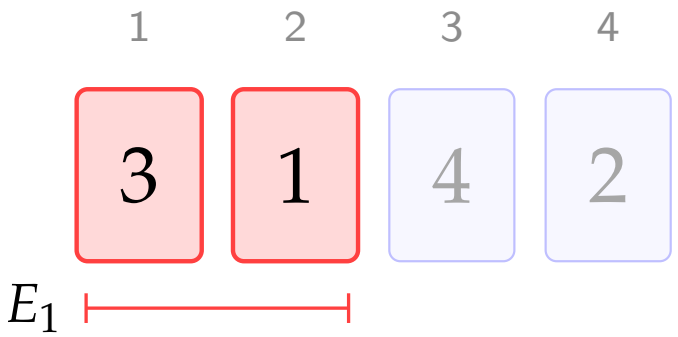
Main tool: E_r

$$E_r f(x) = p_{x_r,x_{r+1}} f(x) + p_{x_{r+1},x_r} f(x\tau_r), \qquad K = \frac{1}{n-1} \sum_{r=1}^{n-1} E_r.$$



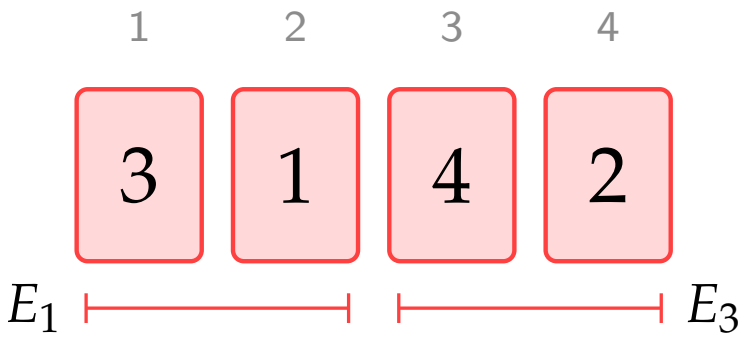
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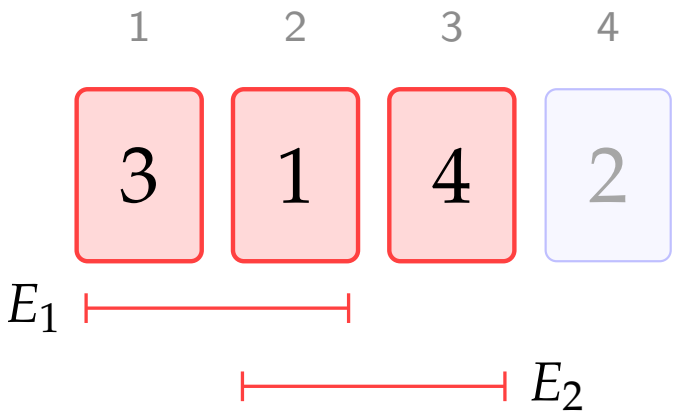
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disjoint:
$E_1 E_3 = E_3 E_1$

Main tool: E_r

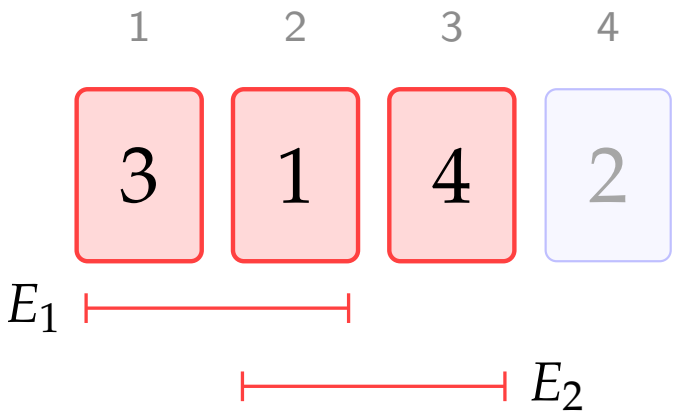
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overlap:
three cards

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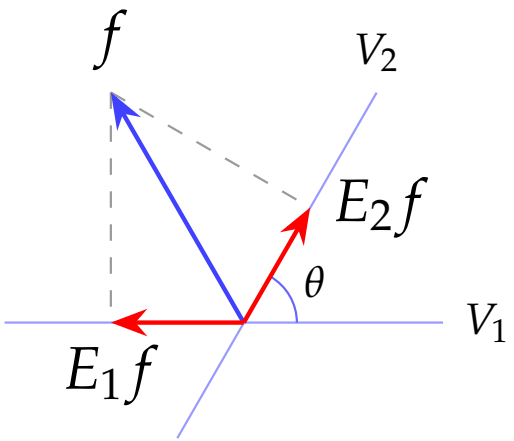


overlap:
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Observation: With $\langle f, g\rangle := \sum_x f(x)\, g(x)\, \mu(x)$, each E_r is the orthogonal projection onto $V_r := \{f : f(x\tau_r) = f(x) \text{ for all } x\}$. So K is an average of projections.

Three cards, again

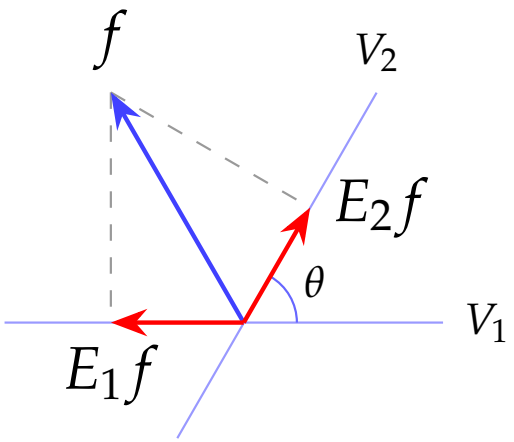
$n = 3$: $K = \frac{1}{2}(E_1 + E_2)$ is the average of two orthogonal projections.



- Two lines at angle θ: $E_1 + E_2$ has eigenvalues $1 \pm \cos\theta$.

Three cards, again

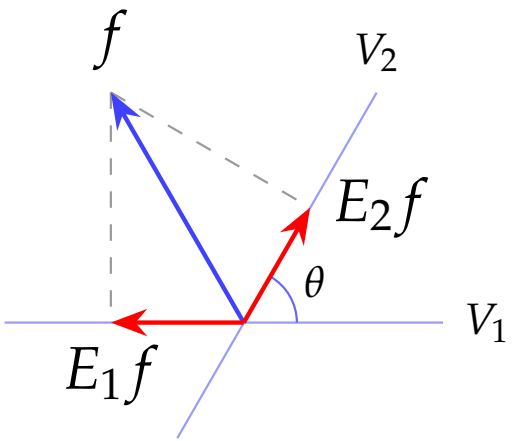
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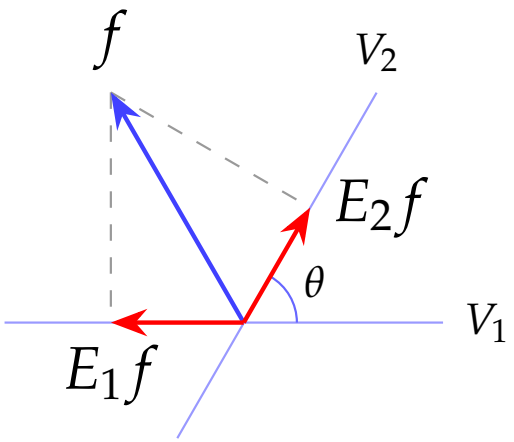
Lemma: For all f and r,

$$\langle E_r f, E_{r+1} f \rangle \geqslant -m_p \, \|E_r f\| \, \|E_{r+1} f\|.$$

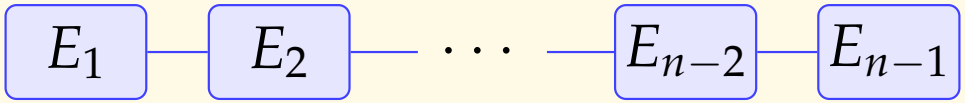
$E_r + E_{r+1}$ only sees three cards, so its smallest positive eigenvalue is $1 - m_p$.

Three cards, again

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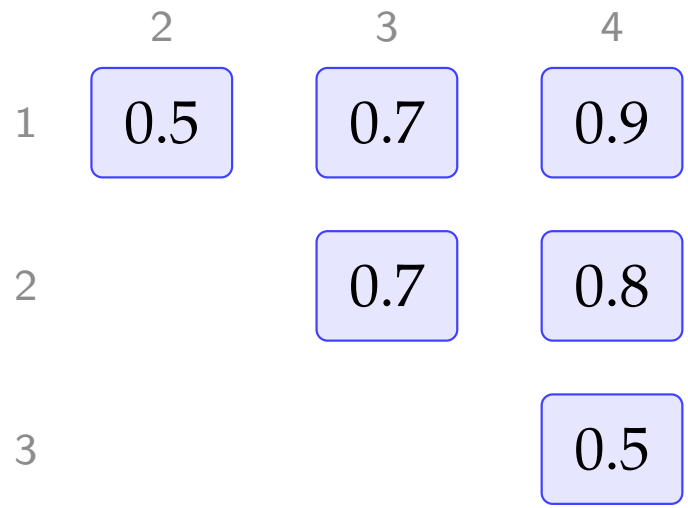
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Only neighbouring projections can give negative cross terms.
The path P_{n-1} has largest adjacency eigenvalue $2\cos(\pi/n)$, giving the main bound.

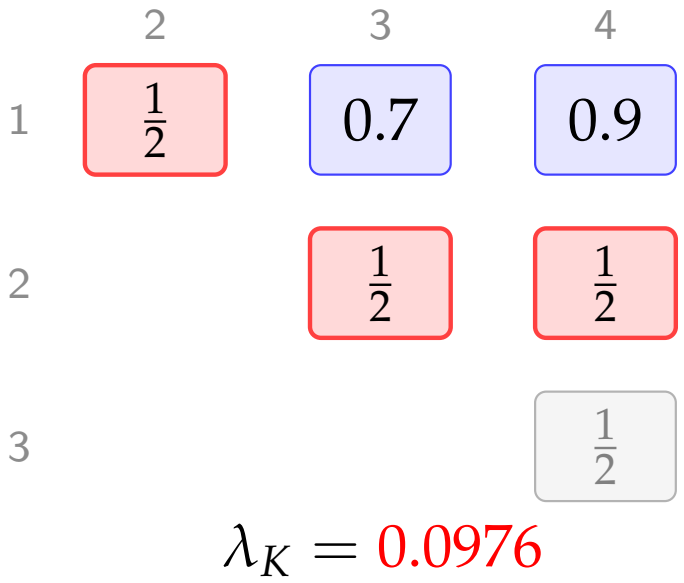
Extremal chains

Greaves and Zhu (2026): Let $n \geqslant 3$ and let p be regular. Then $\lambda_K = \frac{1-\cos(\pi/n)}{n-1}$ if and only if some card i is neutral: $p_{i,j} = \frac{1}{2}$ for all $j \neq i$. Conjectured by Fill.



Extremal chains

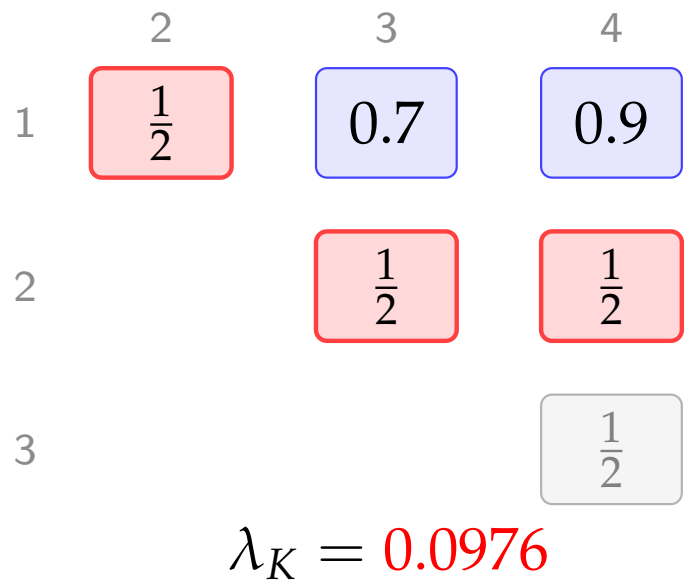
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- Make card 2 neutral: λ_K hits the minimum 0.0976.

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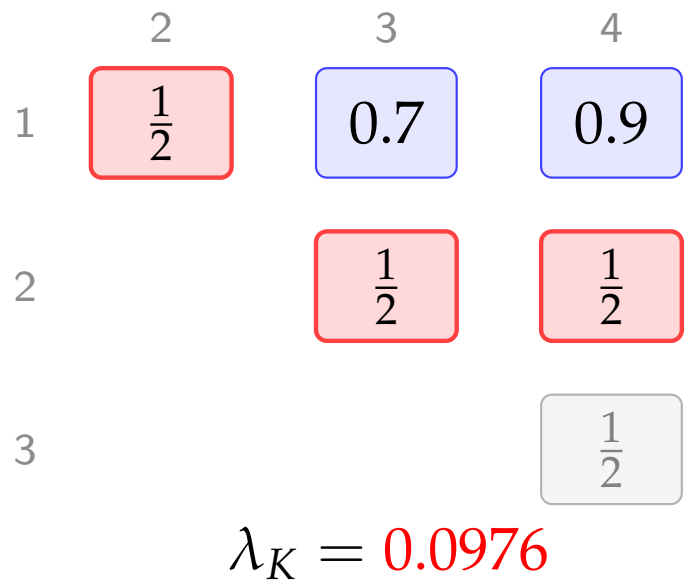
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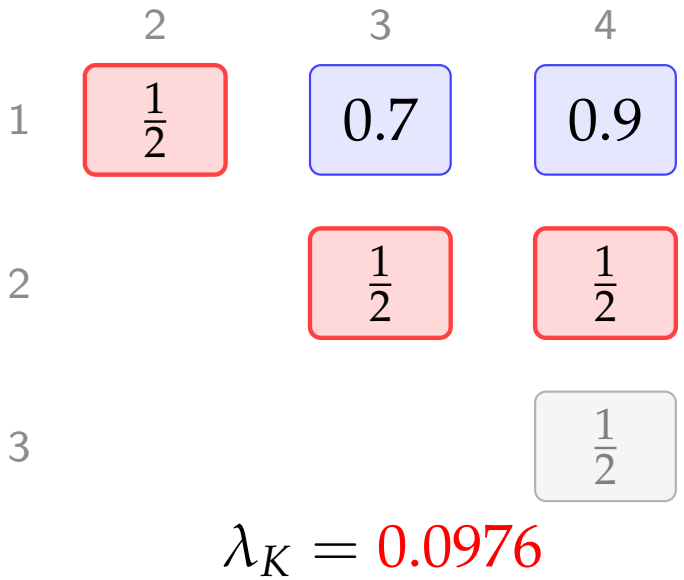


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- ($\Leftarrow$) card i walks as in the unbiased chain.
- ($\Rightarrow$) equality in every step of the proof forces a neutral card.

Multiplicity of the second-largest eigenvalue

Greaves and Zhu (2026): Let $n \geqslant 3$, let p be regular with minimum spectral gap, and let ν_p be the number of neutral cards. Then

$$\operatorname{mult}_K(1-\lambda_K) = \begin{cases} \nu_p, & \nu_p \notin \{n-2, n\}, \\ n-1, & \nu_p \in \{n-2, n\}. \end{cases}$$

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Conjectured by Fill.

neutral cards ($n=4$)	ν_p	multiplicity
$\{2\}$	1	1
$\{3\}$	1	1
$\{2,3\}$	2	3
$\{1,2,3,4\}$	4	3

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$p \equiv \frac{1}{2}$ **(the permutahedron):** Bacher (1994), Wilson (2004): multiplicity $\geqslant n-1$, via Wilson's one-card eigenvectors $f_1, \ldots, f_n$.

Multiplicity of the second-largest eigenvalue

Greaves and Zhu (2026): Let $n \geqslant 3$, let p be regular with minimum spectral gap, and let ν_p be the number of neutral cards. Then

$$\mathrm{mult}_K(1-\lambda_K) = \begin{cases} \nu_p, & \nu_p \notin \{n-2, n\}, \\ n-1, & \nu_p \in \{n-2, n\}. \end{cases}$$

Conjectured by Fill.

$p \equiv \frac{1}{2}$ **(the permutahedron):** Bacher (1994), Wilson (2004): multiplicity $\geqslant n-1$, via Wilson's one-card eigenvectors $f_1, \ldots, f_n$.

Now: exactly $n-1$. So the second eigenspace is $\mathrm{span}\{f_1, \ldots, f_n\}$: the standard representation of S_n, and no other.

Thanks for listening!

arXiv:2603.26303

Backup: two families

n	$p \equiv \frac{1}{2}$	$p_{i,j} = 0.7 \quad (i < j)$ bound	λ_K	$w_i = 2^{-i}$ bound	λ_K
3	0.2500	0.2709	0.2709	0.2892	0.2892
4	0.0976	0.1173	0.1173	0.1346	0.1401
5	0.0477	0.0646	0.0646	0.0794	0.0909
6	0.0268	0.0413	0.0413	0.0539	0.0678
7	0.0165	0.0290	0.0290	0.0400	0.0550

Note: for constant bias the bound is attained for every n: card 1 alone does a biased random walk on P_n.

Backup: λ_K is not monotone

	2	3	4
1	0.5	0.7	0.9
2		0.7	0.8
3			0.5

$\lambda_K = 0.1261$

$p_{1,2} \uparrow$

	2	3	4
1	0.6	0.7	0.9
2		0.7	0.8
3			0.5

$\lambda_K = 0.1259$

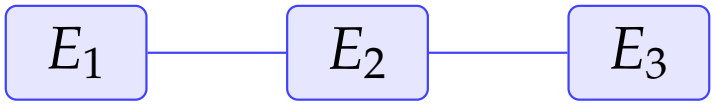
- Fill (2003): λ_K is not monotone in p.
- Fill (2003): nor is every eigenvalue extremal at $p \equiv \frac{1}{2}$: the 5th largest is 0.7887 there, but 0.7944 for another regular p.

Backup: multiplicity for $n = 6$

neutral cards	ν_p	multiplicity
$\{3\}$	1	1
$\{3,4\}$	2	2
$\{2,3,4\}$	3	3
$\{2,3,4,5\}$	4	5
$\{1,\ldots,6\}$	6	5

Backup: Four cards

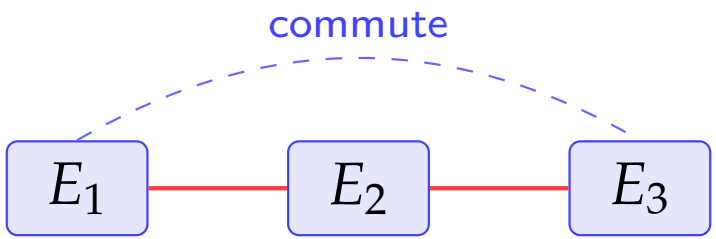
Let $Kf = \theta f$ with $\theta > 0$, and put $v_r := \|E_r f\|$.



$$9\,\langle f, K^2 f\rangle = \sum_{r,s} \langle E_r f, E_s f\rangle$$

Backup: Four cards

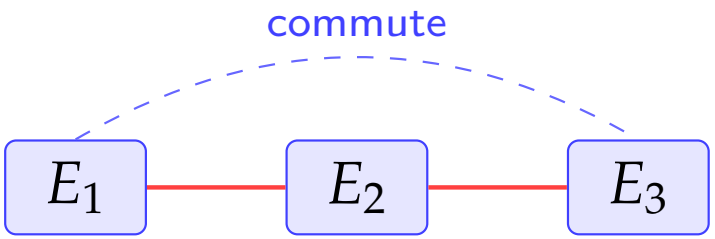
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$$9\,\langle f, K^2 f\rangle = \textstyle\sum_{r,s}\langle E_r f, E_s f\rangle$$
$$\geqslant v_1^2 + v_2^2 + v_3^2 - 2\,m_p\,(v_1 v_2 + v_2 v_3) \qquad \text{(Lemma)}$$

Backup: Four cards

Let $Kf = \theta f$ with $\theta > 0$, and put $v_r := \|E_r f\|$.

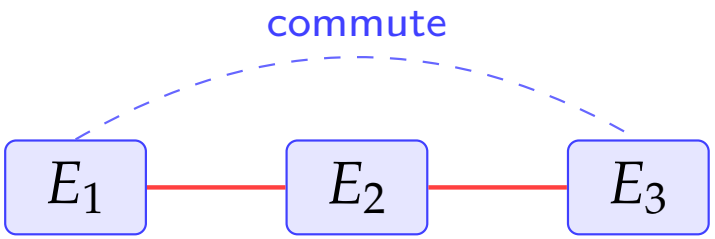


the path P_3, $\lambda_{\max} = \sqrt{2}$

$$
\begin{aligned}
9\,\langle f, K^2 f\rangle &= \textstyle\sum_{r,s}\langle E_r f, E_s f\rangle \\
&\geqslant v_1^2 + v_2^2 + v_3^2 - 2\,m_p\,(v_1 v_2 + v_2 v_3) && \text{(Lemma)} \\
&\geqslant (1 - \sqrt{2}\,m_p)\,(v_1^2 + v_2^2 + v_3^2) && (P_3)
\end{aligned}
$$

Backup: Four cards

Let $Kf = \theta f$ with $\theta > 0$, and put $v_r := \|E_r f\|$.

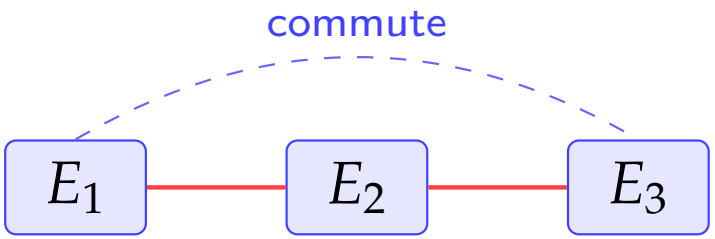


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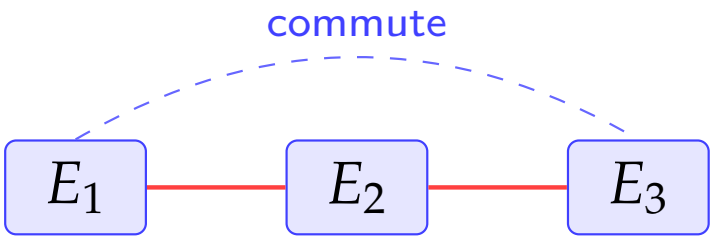
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$\theta \geqslant \frac{1}{3}(1 - \sqrt{2}\,m_p)$, i.e. 0.1173 for $p^\star$.

□

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$\theta \geqslant \frac{1}{3}(1 - \sqrt{2}\,m_p)$, i.e. 0.1173 for $p^\star$.
General n: $P_3 \to P_{n-1}$ and $\sqrt{2} \to 2\cos\frac{\pi}{n}$ give the main result. □